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A class of solutions to the quantum colored Yang–Baxter equation ☆

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Abstract

In this paper, we give a new method to solve the quantum colored Yang–Baxter matrix equation (QCYBE), and a class of solutions to the QCYBE is given.

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Keywords: Yang–Baxter equation; General solution

1. Introduction

Let $\check{R}(u,x,y)$ be an $n^2 \times n^2$ matrix with meromorphic function entries in three independent complex variables u, x and y, and with $\det(\check{R}(u,x,y)) \neq 0$. Here, and throughout this paper, the notation $f(u,x,y) \neq 0$ is used to signify that $f(u,x,y)$ is not the zero function 0, but it may have zero points in $\mathbb{C}^3$. Define

$$\check{R}^{12}(u,x,y) = \check{R}(u,x,y) \otimes E, \qquad \check{R}^{23}(u,x,y) = E \otimes \check{R}(u,x,y),$$

where E is the identity matrix of order n, and $\otimes$ denotes the tensor product of two matrices. By the quantum colored Yang–Baxter equation (QCYBE) we mean the matrix equation

$$\check{R}^{12}(u,x,y)\check{R}^{23}(u+v,x,z)\check{R}^{12}(v,y,z) = \check{R}^{23}(v,y,z)\check{R}^{12}(u+v,x,z)\check{R}^{23}(u,x,y).$$

The QCYBE depends on both the spectral parameters u, v and the color parameters x, y, z; and when it is independent of the color parameters x, y, z, it reduces to the usual YBE (see, e.g. [20,21] and [3,12]), which was proposed independently by Yang [20] and Baxter [3]. Besides its particular importance in mathematical physics, the QCYBE is relevant to many branches of mathematics, such as quantum groups, quantum field theory, lower dimension topology, knot theory, etc. (see, e.g. [1,2,4,6–11,13,14,16,17,22]). For the investigation of QCYBE, one of the most important problems is to give the general solutions of it. For instance, it is known that to give an exact solution of a statistical

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model from physics, the general solution for the corresponding YBE is needed. Also, the general solution of QCYBE is relevant to the linear representation of braid groups, and to the colored Jones polynomials (see, e.g. [17]). So, up to now, a lot of research interest have been paid to find exact solutions for the QCYBE (see, e.g. [5,10,14,18]). And for the most interesting example in physics with $\check{R}(u,x,y)$ being the so-called eight-vertex type of the form

$$\check{R}(u,x,y)=\begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix},$$

many useful results have been achieved (see, e.g. [19]). The purpose of our studies is to illustrate a method of solving the quantum colored Yang–Baxter equation when the quantum colored Yang–Baxter matrix

$$\check{R}(u,x,y)=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \tag{1.1}$$

is upper triangular, where a_{ij} are unknown meromorphic functions in variables u, x and y. So the basic assumption throughout our studies is

$$a_{ij}(u,x,y)=0 \quad \text{for } 1 \leqslant j < i \leqslant 4, \quad \text{and} \quad a_{ii}(u,x,y) \neq 0 \quad \text{for } 1 \leqslant i \leqslant 4. \tag{1.2}$$

We note that our method applies also to the lower triangle case; so our results contain some cases of the eight-vertex type solutions as special cases. For simplification of notations, we always use the symbol

$$p(u,x,y)=a_{44}(u,x,y)-a_{22}(u,x,y)a_{33}(u,x,y). \tag{1.3}$$

We divide our studies into four papers according as $p(u,x,y)$ and $a_{23}(u,x,y)$ are zero functions or not. In the first one (see [15]) of our studies, we considered the case of $a_{23}(u,x,y)p(u,x,y) \neq 0$, and a class of solutions to the QCYBE was given. In this paper, we continue the investigations of [15] to study the case of

$$a_{23}(u,x,y) \neq 0, \quad p(u,x,y)=0. \tag{1.4}$$

Different from that in [15], where $p(u,x,y) \neq 0$, the assumption $p(u,x,y)=0$ of the present paper can be used to reduce the number of independent function equations from the quantum colored Yang–Baxter equation (1.2) of [15], with the a's in (1.1) instead of the $\check{r}$'s there; but, on the other hand, this will cause some special difficulties in solving the remaining equations. This paper will display a method to overcome these difficulties, and give a class of solutions to the quantum colored Yang–Baxter equation under (1.2) and (1.4). Notice that, from the structure of the QCYBE, we can assume without loss of generality that

$$a_{11}(u,x,y)=1.$$

Throughout this paper, we use c and $L(x)$ to denote arbitrary complex constant and arbitrary meromorphic function respectively, they may be different at different occurrences. But the symbols c and $L(x)$ with subscripts will have specified meanings in the context. The main results in this paper are the following Theorems 1–3, which, roughly speaking, guarantee respectively the three cases according to: (i) $a_{22}(u,x,y)a_{33}(u,x,y)=1$; (ii) $a_{22}(u,x,y)a_{33}(u,x,y) \neq 1$, $a_{23}(u,x,y)=1-a_{22}(u,x,y)a_{33}(u,x,y)$; and (iii) $a_{22}(u,x,y)a_{33}(u,x,y) \neq 1$, $a_{23}(u,x,y) \neq 1-a_{22}(u,x,y)a_{33}(u,x,y)$.

Theorem 1. *Assume that $a_{23}(u,x,y) \neq 0$. Then, when all the a_{ij} are transformed by multiplication of meromorphic function so that $a_{11}(u,x,y)=1$, and when $a_{44}(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y)=1$, the general solution of the quantum colored Yang–Baxter equation can be expressed as*

$$a_{22}(u,x,y)=\exp(\alpha_2 u)M_2(x)/M_2(y),$$

$$a_{33}(u,x,y)=\exp(-\alpha_2 u)M_2(y)/M_2(x),$$

$$a_{44}(u,x,y)=1,$$

$$a_{12}(u,x,y) = a_{22}(u,x,y)L(y) - L(x),$$
$$a_{23}(u,x,y) = \alpha u + N(x) - N(y),$$
$$a_{34}(u,x,y) = L(y) - a_{33}(u,x,y)L(x),$$
$$a_{13}(u,x,y) = a_{33}(u,x,y)L(x) - L(y) + a_{23}(u,x,y)L(y),$$
$$a_{24}(u,x,y) = L(x) - a_{23}(u,x,y)L(x) - a_{22}(u,x,y)L(y),$$
$$a_{14}(u,x,y) = 2L(x)L(y) - a_{23}(u,x,y)L(x)L(y) - a_{33}(u,x,y)L(x)L(x) - a_{22}(u,x,y)L(y)L(y),$$

where α and α_2 are arbitrary complex constants, $M_2(x) \neq 0$, $L(x)$ and $N(x)$ are arbitrary meromorphic functions such that $N(x)$ is not a constant when $\alpha = 0$.

Theorem 2. *Assume that $a_{23}(u,x,y) \neq 0$. Then, when all the a_{ij} are transformed by multiplication of meromorphic function so that $a_{11}(u,x,y) = 1$, and when $a_{44}(u,x,y) = a_{22}(u,x,y)a_{33}(u,x,y) \neq 1$, $a_{23}(u,x,y) = 1 - a_{22}(u,x,y)a_{33}(u,x,y)$, the general solution of the quantum colored Yang–Baxter equation can be expressed as*

$$a_{22}(u,x,y) = \exp(\alpha_2 u)M_2(x)/M_2(y),$$
$$a_{33}(u,x,y) = \exp(\alpha_3 u)M_3(x)/M_3(y),$$
$$a_{44}(u,x,y) = a_{22}(u,x,y)a_{33}(u,x,y),$$
$$a_{12}(u,x,y) = L(x) - a_{22}(u,x,y)L(y),$$
$$a_{23}(u,x,y) = 1 - a_{22}(u,x,y)a_{33}(u,x,y),$$
$$a_{34}(u,x,y) = a_{33}(u,x,y)a_{12}(u,x,y),$$
$$a_{13}(u,x,y) = -a_{34}(u,x,y),$$
$$a_{24}(u,x,y) = 2a_{23}(u,x,y)L(x) - a_{12}(u,x,y),$$
$$a_{14}(u,x,y) = c^2 M_2(x)M_2(y)a_{23}(u,x,y) - a_{33}(u,x,y)L(x)^2 - a_{22}(u,x,y)L(y)^2 + 2a_{22}(u,x,y)a_{33}(u,x,y)L(x)L(y),$$

where α_2, α_3 and c are complex constants satisfying $c\alpha_2 = 0$, and $M_2(x) \neq 0$, $M_3(x) \neq 0$ and $L(x)$ are arbitrary meromorphic functions such that $M_2(x)M_3(x)$ is not a constant when $\alpha_2 + \alpha_3 = 0$.

Theorem 3. *Assume that $a_{23}(u,x,y) \neq 0$. Then, when all the a_{ij} are transformed by multiplication of meromorphic function so that $a_{11}(u,x,y) = 1$, and when $a_{44}(u,x,y) = a_{22}(u,x,y)a_{33}(u,x,y) \neq 1$, $a_{23}(u,x,y) \neq 1 - a_{22}(u,x,y)a_{33}(u,x,y)$, the general solution of the quantum colored Yang–Baxter equation can be expressed as*

$$a_{22}(u,x,y) = \exp(\alpha_2 u)M_2(x)/M_2(y),$$
$$a_{33}(u,x,y) = \exp(\alpha_3 u)M_3(x)/M_3(y),$$
$$a_{44}(u,x,y) = a_{22}(u,x,y)a_{33}(u,x,y),$$
$$a_{12}(u,x,y) = L(x) - a_{22}(u,x,y)L(y),$$
$$a_{34}(u,x,y) = a_{33}(u,x,y)a_{12}(u,x,y),$$
$$a_{13}(u,x,y) = L(y) - a_{23}(u,x,y)L(y) - a_{33}(u,x,y)L(x),$$
$$a_{24}(u,x,y) = a_{22}(u,x,y)L(y) + a_{23}(u,x,y)L(x) - a_{22}(u,x,y)a_{33}(u,x,y)L(x),$$
$$a_{14}(u,x,y) = ca_{23}(u,x,y)M_2(x)M_2(y) - a_{33}(u,x,y)L(x)^2 - a_{23}(u,x,y)L(x)L(y) + L(x)L(y) + a_{22}(u,x,y)a_{33}(u,x,y)L(x)L(y) - a_{22}(u,x,y)L(y)^2,$$

$$a_{23}(u,x,y)=\begin{cases} M(x)-a_{22}(u,x,y)a_{33}(u,x,y)M(y)+\lambda[1-\exp((\alpha_2+\alpha_3)u)]M_2(x)M_3(x) \\ \quad +\lambda\delta u M_2(x)M_3(x), \quad \text{if } c=0, \\ a_{22}(u,x,y)a_{33}(u,x,y)\sqrt{1-\mu M_2(y)^2M_3(y)^2}-\sqrt{1-\mu M_2(x)^2M_3(x)^2}, \quad \text{if } c\neq 0. \end{cases}$$

Here $\delta=1$ or 0 according to $\alpha_2+\alpha_3=0$ or not; α_2, α_3, c, λ and μ are arbitrary complex constants satisfying $\mu(\alpha_2+\alpha_3)=0$, $c\alpha_2=0$; $L(x)$, $M(x)$, $M_2(x)\neq 0$ and $M_3(x)\neq 0$ are arbitrary meromorphic functions satisfying one of the following conditions:

(1) If $c\neq 0$ and $\alpha_2+\alpha_3=0$, then $M_2(x)M_3(x)$ is not a constant.
(2) If $c=0$ and $\alpha_2+\alpha_3\neq 0$, then $M(x)\neq -\lambda M_2(x)M_3(x)$, $M(x)\neq 1-\lambda M_2(x)M_3(x)$.
(3) If $c=0$ and $\alpha_2+\alpha_3=0$, then $M_2(x)M_3(x)$ is not a constant, and when $\lambda=0$, both $\frac{M(x)}{M_2(x)M_3(x)}$ and $\frac{M(x)-1}{M_2(x)M_3(x)}$ are not constants.

The material of this paper is arranged as follows. In Section 2 we give a classification of the system of the nonlinear function equations from the quantum colored Yang–Baxter equation. Then in Section 3 we investigate the first 9 equations in categories (B1) to (B3) below. It forms a basis for our further discussions. In Sections 4 and 5 we give the proofs of Theorems 1 and 2 respectively. Section 6 is served to exclude the third case in Lemma 3.2. Finally, in Section 7, we accomplish the proof of Theorems 3, and at the same time, we give a lemma which seems to have some general interest and is used in the proof of Theorem 3.

Remark. The QCYBE clearly has a trivial solution with all of the unknown functions to be 0. But, it seems difficult to give nontrivial solutions in general. However, using the above theorems this can be done easily. For instance, if we take $c=\lambda=\mu=L(x)=0$, $\alpha_2=1$, $\alpha_3=-1$, $M(x)=2$, $M_2(x)=x$, $M_3(x)=x^2$ in Theorem 3, then all the conditions are satisfied, and we can give a solution of the QCYBE as follows:

$$a_{11}(u,x,y)=1,\quad a_{22}(u,x,y)=\exp(u)xy^{-1},\quad a_{33}(u,x,y)=\exp(-u)x^2y^{-2},$$
$$a_{44}(u,x,y)=x^3y^{-3},\quad a_{23}(u,x,y)=2-2x^3y^{-3},$$
$$a_{12}(u,x,y)=a_{13}(u,x,y)=a_{14}(u,x,y)=a_{24}(u,x,y)=a_{34}(u,x,y)=0.$$

Note that the verification of the validity of the following (E1)–(E22) to this solution is almost trivial.

2. Classification

We know from the discussion in §2 of [15] that there are $2^6=64$ nonlinear function equations in total from the QCYBE, and there are only 24 nontrivial ones from them. Under the present assumption (1.4), the third one (E3) and the seventh one (E7) in [15] are consequences of (E1), (E2) and (E5) there. So when (E1), (E2) and (E5) in [15] are taken into consideration, then (E3) and (E7) there become redundant. Thus, based on the results in §2 of [15], the QCYBE is now reduced to the following 22 nontrivial equations (E1) to (E22), which can be grouped into the following five categories (B1) to (B5). Note that, in our classification, the function equations in the former category often contain fewer unknown functions than that in the latter. So, logically, the principal idea of solving the system of the 22 equations is that we first consider the simple ones containing fewer unknown functions to get general solutions for some unknown functions, then with the known solutions for the solved unknown functions in mind, we come to consider the remaining equations which often contain more unknown functions. In the process, we keep in mind that the solutions of the solved unknown functions must satisfy the remaining equations besides those used to induce the solutions.

(B1) The function equations only with diagonal unknowns:

$$a_{22}(u+v,x,z)=a_{22}(u,x,y)a_{22}(v,y,z), \tag{E1}$$
$$a_{33}(u+v,x,z)=a_{33}(u,x,y)a_{33}(v,y,z). \tag{E2}$$

(B2) The function equations only with unknowns on the diagonal and on the line a_{12}, a_{23} and a_{34} in the matrix of (1.1):

$$a_{12}(u+v,x,z)=a_{12}(u,x,y)+a_{22}(u,x,y)a_{12}(v,y,z), \tag{E3}$$

$$a_{23}(u+v,x,z)=a_{23}(u,x,y)+a_{22}(u,x,y)a_{33}(u,x,y)a_{23}(v,y,z), \tag{E4}$$

$$a_{34}(u+v,x,z)=a_{33}(v,y,z)a_{34}(u,x,y)+a_{22}(u,x,y)a_{33}(u,x,y)a_{34}(v,y,z), \tag{E5}$$

$$\begin{aligned}&a_{33}(v,y,z)a_{23}(u+v,x,z)a_{12}(u,x,y)+a_{22}(u,x,y)a_{23}(u+v,x,z)a_{34}(v,y,z)\\&\quad=a_{33}(v,y,z)a_{23}(u,x,y)a_{12}(u+v,x,z)+a_{22}(u,x,y)a_{23}(v,y,z)a_{34}(u+v,x,z).\end{aligned} \tag{E6}$$

(B3) The function equations with unknown a_{13} and the unknowns in (B2):

$$\begin{aligned}&a_{13}(u+v,x,z)-a_{23}(v,y,z)a_{13}(u+v,x,z)\\&\quad=a_{33}(v,y,z)a_{13}(u,x,y)+a_{13}(v,y,z)-a_{23}(u+v,x,z)a_{13}(v,y,z)\\&\qquad+a_{33}(u+v,x,z)a_{23}(v,y,z)a_{12}(u,x,y),\end{aligned} \tag{E7}$$

$$\begin{aligned}&a_{22}(v,y,z)a_{13}(u+v,x,z)\\&\quad=a_{13}(u,x,y)+a_{22}(u+v,x,z)a_{33}(u,x,y)a_{13}(v,y,z)\\&\qquad-a_{22}(v,y,z)a_{33}(u+v,x,z)a_{12}(u,x,y)-a_{12}(v,y,z)\\&\qquad+a_{33}(u,x,y)a_{12}(u+v,x,z)+a_{23}(u,x,y)a_{12}(v,y,z),\end{aligned} \tag{E8}$$

$$\begin{aligned}&a_{22}(v,y,z)a_{33}(v,y,z)a_{13}(u+v,x,z)\\&\quad=a_{33}(v,y,z)a_{13}(u,x,y)+a_{22}(u+v,x,z)a_{33}(u+v,x,z)a_{13}(v,y,z)\\&\qquad+a_{33}(v,y,z)a_{23}(u,x,y)a_{12}(v,y,z)-a_{34}(v,y,z)+a_{34}(u+v,x,z)\\&\qquad-a_{22}(v,y,z)a_{33}(v,y,z)a_{33}(v,y,z)a_{34}(u,x,y).\end{aligned} \tag{E9}$$

(B4) The function equations with unknowns a_{13}, a_{24} and the unknowns in (B2):

$$\begin{aligned}&a_{24}(u+v,x,z)\\&\quad=a_{24}(u,x,y)+a_{22}(u,x,y)a_{22}(u,x,y)a_{33}(u,x,y)a_{24}(v,y,z)\\&\qquad+a_{22}(u,x,y)a_{34}(u,x,y)a_{23}(v,y,z)-a_{22}(u,x,y)a_{12}(v,y,z)\\&\qquad+a_{22}(u,x,y)a_{33}(u,x,y)a_{12}(u+v,x,z)\\&\qquad-a_{22}(u+v,x,z)a_{33}(u+v,x,z)a_{12}(u,x,y),\end{aligned} \tag{E10}$$

$$\begin{aligned}&a_{33}(v,y,z)a_{24}(u+v,x,z)\\&\quad=a_{33}(v,y,z)a_{24}(u,x,y)+a_{22}(u,x,y)^2a_{33}(u+v,x,z)a_{24}(v,y,z)\\&\qquad+a_{22}(u,x,y)a_{34}(u+v,x,z)-a_{22}(u,x,y)a_{34}(v,y,z)\\&\qquad+a_{22}(u,x,y)a_{33}(v,y,z)a_{23}(v,y,z)a_{34}(u,x,y)\\&\qquad-a_{33}(v,y,z)a_{22}(u+v,x,z)a_{33}(v,y,z)a_{34}(u,x,y),\end{aligned} \tag{E11}$$

$$\begin{aligned}&a_{33}(v,y,z)a_{23}(u,x,y)a_{24}(u+v,x,z)\\&\qquad-a_{22}(u,x,y)a_{33}(u+v,x,z)a_{24}(u+v,x,z)\\&\quad=a_{33}(v,y,z)a_{23}(u+v,x,z)a_{24}(u,x,y)\\&\qquad-a_{33}(v,y,z)a_{22}(u+v,x,z)a_{33}(u+v,x,z)a_{24}(u,x,y)\\&\qquad-a_{22}(u,x,y)^2a_{33}(u+v,x,z)a_{24}(v,y,z)\\&\qquad-a_{22}(u,x,y)a_{23}(u,x,y)a_{34}(v,y,z),\end{aligned} \tag{E12}$$

$$\begin{aligned}&a_{22}(u+v,x,z)a_{13}(u+v,x,z)-a_{22}(u+v,x,z)a_{13}(v,y,z)\\&\qquad-a_{22}(u+v,x,z)a_{33}(v,y,z)a_{13}(u,x,y)\\&\quad=a_{23}(v,y,z)a_{24}(u+v,x,z)-a_{22}(u,x,y)a_{23}(u+v,x,z)a_{24}(v,y,z)\end{aligned}$$

$$
\begin{aligned}
&\quad - a_{23}(v,y,z)a_{23}(u+v,x,z)a_{12}(u,x,y)\\
&\quad + a_{22}(u+v,x,z)a_{33}(u+v,x,z)a_{23}(v,y,z)a_{12}(u,x,y), && \text{(E13)}\\
&a_{33}(u,x,y)a_{33}(v,y,z)a_{24}(u+v,x,z) - a_{22}(u,x,y)a_{33}(u+v,x,z)a_{24}(v,y,z)\\
&\quad - a_{22}(v,y,z)a_{33}(v,y,z)a_{33}(u+v,x,z)a_{24}(u,x,y)\\
&= a_{23}(u,x,y)a_{34}(v,y,z) - a_{33}(v,y,z)a_{23}(u+v,x,z)a_{13}(u,x,y)\\
&\quad + a_{22}(v,y,z)a_{33}(v,y,z)a_{23}(u,x,y)a_{13}(u+v,x,z)\\
&\quad - a_{23}(u,x,y)a_{34}(v,y,z)a_{23}(u+v,x,z), && \text{(E14)}\\
&a_{33}(u,x,y)a_{23}(v,y,z)a_{24}(u+v,x,z) - a_{23}(u+v,x,z)a_{24}(v,y,z)\\
&\quad + a_{23}(u,x,y)a_{23}(u+v,x,z)a_{24}(v,y,z)\\
&= -a_{22}(v,y,z)a_{23}(u,x,y)a_{13}(u+v,x,z) - a_{23}(v,y,z)a_{23}(u+v,x,z)a_{13}(u,x,y)\\
&\quad + a_{22}(v,y,z)a_{33}(v,y,z)a_{23}(u+v,x,z)a_{13}(u,x,y). && \text{(E15)}
\end{aligned}
$$

(B5) The remaining ones which all contain the unknown a_{14}:

$$
\begin{aligned}
&a_{23}(v,y,z)a_{14}(u+v,x,z)\\
&= a_{22}(u,x,y)a_{23}(u+v,x,z)a_{14}(v,y,z) + a_{12}(u,x,y)a_{13}(u+v,x,z)\\
&\quad + a_{22}(u,x,y)a_{12}(v,y,z)a_{13}(u+v,x,z) - a_{33}(v,y,z)a_{12}(u+v,x,z)a_{13}(u,x,y)\\
&\quad + a_{23}(u+v,x,z)a_{12}(u,x,y)a_{13}(v,y,z) - a_{12}(u+v,x,z)a_{13}(v,y,z)\\
&\quad - a_{23}(v,y,z)a_{12}(u,x,y)a_{34}(u+v,x,z), && \text{(E16)}\\
&a_{22}(v,y,z)a_{14}(u+v,x,z)\\
&= a_{14}(u,x,y) + a_{22}(u,x,y)a_{33}(u,x,y)a_{22}(u+v,x,z)a_{14}(v,y,z)\\
&\quad + a_{22}(u+v,x,z)a_{34}(u,x,y)a_{13}(v,y,z) + a_{12}(v,y,z)a_{24}(u,x,y)\\
&\quad + a_{34}(u,x,y)a_{12}(u+v,x,z) + a_{22}(u,x,y)a_{33}(u,x,y)a_{12}(v,y,z)a_{12}(u+v,x,z)\\
&\quad - a_{12}(v,y,z)a_{12}(u+v,x,z) - a_{22}(v,y,z)a_{12}(u,x,y)a_{34}(u+v,x,z), && \text{(E17)}\\
&a_{22}(v,y,z)a_{33}(v,y,z)a_{23}(u,x,y)a_{14}(u+v,x,z)\\
&= a_{33}(v,y,z)a_{23}(u+v,x,z)a_{14}(u,x,y) + a_{33}(v,y,z)a_{34}(u,x,y)a_{24}(u+v,x,z)\\
&\quad + a_{22}(u,x,y)a_{33}(u,x,y)a_{34}(v,y,z)a_{24}(u+v,x,z)\\
&\quad + a_{23}(u+v,x,z)a_{34}(v,y,z)a_{24}(u,x,y)\\
&\quad - a_{22}(v,y,z)a_{33}(v,y,z)a_{34}(u+v,x,z)a_{24}(u,x,y)\\
&\quad - a_{22}(u,x,y)a_{34}(u+v,x,z)a_{24}(v,y,z)\\
&\quad - a_{23}(u,x,y)a_{34}(v,y,z)a_{12}(u+v,x,z), && \text{(E18)}\\
&a_{22}(v,y,z)a_{33}(v,y,z)a_{33}(u,x,y)a_{14}(u+v,x,z)\\
&= a_{33}(u+v,x,z)a_{14}(u,x,y) + a_{33}(u+v,x,z)a_{12}(v,y,z)a_{24}(u,x,y)\\
&\quad + a_{22}(u,x,y)a_{33}(u,x,y)a_{22}(u+v,x,z)a_{33}(u+v,x,z)a_{14}(v,y,z)\\
&\quad + a_{22}(u+v,x,z)a_{33}(u+v,x,z)a_{34}(u,x,y)a_{13}(v,y,z)\\
&\quad + a_{34}(u,x,y)a_{34}(u+v,x,z) - a_{33}(u,x,y)a_{34}(v,y,z)a_{12}(u+v,x,z)\\
&\quad - a_{22}(v,y,z)a_{33}(v,y,z)a_{34}(u,x,y)a_{34}(u+v,x,z)\\
&\quad + a_{22}(u,x,y)a_{33}(u,x,y)a_{12}(v,y,z)a_{34}(u+v,x,z), && \text{(E19)}\\
&a_{33}(u,x,y)a_{14}(u+v,x,z)\\
&= a_{22}(v,y,z)a_{33}(v,y,z)a_{33}(u+v,x,z)a_{14}(u,x,y)
\end{aligned}
$$

$$
\begin{aligned}
&- a_{23}(u,x,y)a_{23}(u+v,x,z)a_{14}(v,y,z)\\
&+ a_{14}(v,y,z) + a_{22}(v,y,z)a_{33}(v,y,z)a_{13}(u,x,y)a_{13}(u+v,x,z)\\
&+ a_{24}(v,y,z)a_{13}(u+v,x,z) - a_{13}(u,x,y)a_{13}(u+v,x,z)\\
&- a_{23}(u,x,y)a_{12}(v,y,z)a_{13}(u+v,x,z) + a_{34}(v,y,z)a_{13}(u,x,y)\\
&- a_{23}(u+v,x,z)a_{13}(u,x,y)a_{13}(v,y,z)\\
&- a_{33}(u,x,y)a_{13}(v,y,z)a_{24}(u+v,x,z)\\
&+ a_{33}(u+v,x,z)a_{12}(u,x,y)a_{24}(v,y,z),
\end{aligned}
\tag{E20}
$$

$$
\begin{aligned}
&a_{22}(u,x,y)a_{33}(u,x,y)a_{22}(v,y,z)a_{14}(u+v,x,z)\\
&\quad = a_{22}(v,y,z)a_{33}(v,y,z)a_{22}(u+v,x,z)a_{33}(u+v,x,z)a_{14}(u,x,y)\\
&\qquad - a_{23}(v,y,z)a_{23}(u+v,x,z)a_{14}(u,x,y) + a_{22}(u+v,x,z)a_{14}(v,y,z)\\
&\qquad - a_{22}(v,y,z)a_{24}(u,x,y)a_{13}(u+v,x,z) + a_{22}(u+v,x,z)a_{34}(v,y,z)a_{13}(u,x,y)\\
&\qquad + a_{22}(v,y,z)a_{33}(v,y,z)a_{13}(u,x,y)a_{24}(u+v,x,z)\\
&\qquad - a_{23}(v,y,z)a_{34}(u,x,y)a_{24}(u+v,x,z)\\
&\qquad - a_{22}(u,x,y)a_{33}(u,x,y)a_{24}(v,y,z)a_{24}(u+v,x,z) + a_{24}(v,y,z)a_{24}(u+v,x,z)\\
&\qquad + a_{22}(u+v,x,z)a_{33}(u+v,x,z)a_{12}(u,x,y)a_{24}(v,y,z)\\
&\qquad - a_{23}(u+v,x,z)a_{24}(u,x,y)a_{24}(v,y,z),
\end{aligned}
\tag{E21}
$$

$$
\begin{aligned}
&a_{34}(u,x,y)a_{14}(u+v,x,z) + a_{22}(u,x,y)a_{33}(u,x,y)a_{12}(v,y,z)a_{14}(u+v,x,z)\\
&\qquad - a_{24}(v,y,z)a_{14}(u+v,x,z) - a_{22}(v,y,z)a_{33}(v,y,z)a_{13}(u,x,y)a_{14}(u+v,x,z)\\
&\quad = a_{22}(v,y,z)a_{33}(v,y,z)a_{34}(u+v,x,z)a_{14}(u,x,y) - a_{13}(u+v,x,z)a_{14}(u,x,y)\\
&\qquad - a_{13}(v,y,z)a_{23}(u+v,x,z)a_{14}(u,x,y) - a_{23}(u+v,x,z)a_{24}(u,x,y)a_{14}(v,y,z)\\
&\qquad - a_{22}(u,x,y)a_{33}(u,x,y)a_{24}(u+v,x,z)a_{14}(v,y,z) + a_{12}(u+v,x,z)a_{14}(v,y,z)\\
&\qquad + a_{12}(u+v,x,z)a_{34}(v,y,z)a_{13}(u,x,y) - a_{12}(v,y,z)a_{24}(u,x,y)a_{13}(u+v,x,z)\\
&\qquad - a_{34}(u,x,y)a_{13}(v,y,z)a_{24}(u+v,x,z) + a_{12}(u,x,y)a_{34}(u+v,x,z)a_{24}(v,y,z).
\end{aligned}
\tag{E22}
$$

3. The equation systems (B1), (B2) and (B3)

The treatment of (B1) is exactly the same as that in Lemma 2.1 of [15]; the result is given by the following

Lemma 3.1. *The general solution of system* (B1) *of the function equations can be expressed as*

$$a_{22}(u,x,y) = \exp(\alpha_2 u)M_2(x)/M_2(y), \quad a_{33}(u,x,y) = \exp(\alpha_3 u)M_3(x)/M_3(y),$$

where $M_2(x) \neq 0$, $M_3(x) \neq 0$ are arbitrary meromorphic functions of the complex variable $x \in \mathbb{C}$, and α_2, α_3 are arbitrary complex constants.

So from now on, we always assume that $a_{22}(u,x,y)$ and $a_{33}(u,x,y)$ have the explicit form as in this lemma, with α_2 and α_3 parameters, and $M_2(x) \neq 0$ and $M_3(x) \neq 0$ parameter functions. These are not always written explicitly instead of $a_{22}(u,x,y)$ and $a_{33}(u,x,y)$ in the following discussion for the sake of simplicity of text; but we will use the explicit form freely when it is necessary. Similar usages may be applied to other unknown functions when the unknowns are solved.

Now we come to consider (B2) and (B3), with a_{22} and a_{33} being of the form as in Lemma 3.1. Note that the unknown functions now are a_{12}, a_{23}, a_{34} and a_{13}; so our purpose is to give the general form of these unknown functions satisfying equations (E3)–(E9) for the time being, e.g. a_{12} must satisfy (E3), (E6), (E8) and (E9). To solve an unknown, e.g. a_{12}, we first consider the combination of two relatively simple ones, e.g. (E3) and (E6), to give a general form of a_{12}, which often contains some parameters, then we come to check the validity of (E8) and (E9) to this general form of a_{12}. In the process of verification, some additional conditions often appear, which shows that the parameters are not independent from each other, and so we need to find the relationship of the parameters and to determine the independent ones to give a general solution. Here, it is important to note that, in the process of solving one unknown function, some other unknown functions are often twisted together because almost all equations from (E1)–(E22) contain more than one unknown functions with different types of variables (u,x,y), (v,y,z) and $(u+v,x,z)$. Thus, in practice, we often need to consider simultaneously more than one unknown functions at a time. Our result is the following

Lemma 3.2. *Assume* (1.2) *and* (1.4). *Let $a_{22}(u,x,y)$ and $a_{33}(u,x,y)$ be as in Lemma* 3.1. *Then the solutions of systems* (B2) *and* (B3) *of function equations can be formulated as follows*:

(I) *If $a_{44}(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y)=1$, then the general solution can be expressed as*

$$\begin{aligned}
&a_{12}(u,x,y)=a_{22}(u,x,y)L(y)-L(x),\\
&a_{23}(u,x,y)=\alpha u+N(x)-N(y),\\
&a_{34}(u,x,y)=L(y)-a_{33}(u,x,y)L(x)-ca_{23}(u,x,y)M_3(y)^{-1},\\
&a_{13}(u,x,y)=a_{33}(u,x,y)L(x)+a_{23}(u,x,y)L(y)-L(y),
\end{aligned}$$

where α and c are complex constants such that $c\alpha_3=0$ with α_3 as in Lemma 3.1, *$L(x)$ and $N(x)$ are meromorphic functions.*

(II) *If $a_{44}(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y)\neq 1$, $a_{23}(u,x,y)=1-a_{22}(u,x,y)a_{33}(u,x,y)$, then systems* (B2) *and* (B3) *are equivalent to the system*

$$\begin{aligned}
a_{12}(u+v,x,z)&= a_{12}(u,x,y)+a_{22}(u,x,y)a_{12}(v,y,z),\\
a_{34}(u,x,y)&= a_{33}(u,x,y)a_{12}(u,x,y)-ca_{23}(u,x,y)M_3(y)^{-1},\\
a_{22}(v,y,z)a_{13}(u+v,x,z)&=a_{13}(u,x,y)+a_{22}(u+v,x,z)a_{33}(u,x,y)a_{13}(v,y,z)\\
&\quad+a_{33}(u,x,y)a_{23}(v,y,z)a_{12}(u,x,y),
\end{aligned}$$

where c is a complex constant such that $c\alpha_3=0$ with α_3 as in Lemma 3.1.

(III) *If $a_{44}(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y)\neq 1$, $a_{23}(u,x,y)\neq 1-a_{22}(u,x,y)a_{33}(u,x,y)$ and $a_{34}(u,x,y)\neq a_{33}(u,x,y)a_{12}(u,x,y)$, then the general solution can be expressed as*

$$\begin{aligned}
&a_{12}(u,x,y)=L(x)-a_{22}(u,x,y)L(y),\\
&a_{23}(u,x,y)=c'\big[1-a_{22}(u,x,y)a_{33}(u,x,y)\big],\\
&a_{34}(u,x,y)=a_{33}(u,x,y)a_{12}(u,x,y)-ca_{23}(u,x,y)M_3(y)^{-1},\\
&a_{13}(u,x,y)=L(y)-a_{23}(u,x,y)L(y)-a_{33}(u,x,y)L(x),
\end{aligned}$$

where $c\neq 0$ and $c'\neq 0,1$ are two complex constants, $L(x)$ is a meromorphic function, and the α_3 in Lemma 3.1 *is $\alpha_3=0$.*

(IV) *If $a_{44}(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y)\neq 1$, $a_{23}(u,x,y)\neq 1-a_{22}(u,x,y)a_{33}(u,x,y)$ and $a_{34}(u,x,y)=a_{33}(u,x,y)a_{12}(u,x,y)$, then systems* (B2) *and* (B3) *are equivalent to the system*

$$\begin{aligned}
&a_{12}(u,x,y)=L(x)-a_{22}(u,x,y)L(y),\\
&a_{23}(u+v,x,z)=a_{23}(u,x,y)+a_{22}(u,x,y)a_{33}(u,x,y)a_{23}(v,y,z),\\
&a_{13}(u,x,y)=L(y)-a_{23}(u,x,y)L(y)-a_{33}(u,x,y)L(x),
\end{aligned}$$

where $L(x)$ is a meromorphic function.

Proof. We first consider the case

$$a_{44}(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y)=1.$$

Using $a_{22}(u,x,y)a_{33}(u,x,y)=1$, by (E3) and (E5), we can write (E6) as

$$\begin{aligned}&\big[a_{12}(u,x,y)-a_{22}(u,x,y)a_{34}(u,x,y)\big]a_{33}(v,y,z)a_{23}(v,y,z)\\&\quad=a_{22}(u,x,y)a_{23}(u,x,y)\big[a_{33}(v,y,z)a_{12}(v,y,z)-a_{34}(v,y,z)\big].\end{aligned}\tag{3.1}$$

This together with $a_{23}(u,x,y)\neq 0$ implies that there exists a meromorphic function $L_1(y)$ such that

$$a_{12}(u,x,y)=a_{22}(u,x,y)a_{34}(u,x,y)+a_{22}(u,x,y)a_{23}(u,x,y)L_1(y),\tag{3.2}$$

and

$$a_{33}(v,y,z)a_{12}(v,y,z)-a_{34}(v,y,z)=a_{33}(v,y,z)a_{23}(v,y,z)L_1(y).$$

Rewriting the last equation in terms of (u,x,y), we get

$$a_{33}(u,x,y)a_{12}(u,x,y)-a_{34}(u,x,y)=a_{33}(u,x,y)a_{23}(u,x,y)L_1(x).\tag{3.3}$$

The combination of (3.2), (3.3) and $a_{22}(u,x,y)a_{33}(u,x,y)=1$ implies that

$$L_1(x)a_{33}(u,x,y)=L_1(y).\tag{3.4}$$

By (E4) and (3.2), we can write (E7) as

$$\begin{aligned}&\big[a_{13}(u,x,y)+a_{33}(u,x,y)a_{12}(u,x,y)\big]a_{23}(v,y,z)\\&\quad=a_{23}(u,x,y)\big[a_{22}(v,y,z)a_{13}(v,y,z)+a_{12}(v,y,z)-a_{23}(v,y,z)a_{12}(v,y,z)\big].\end{aligned}\tag{3.5}$$

This implies that there exists a meromorphic function $L(y)$ such that

$$a_{13}(u,x,y)=a_{23}(u,x,y)L(y)-a_{33}(u,x,y)a_{12}(u,x,y),\tag{3.6}$$

and

$$a_{22}(v,y,z)a_{13}(v,y,z)+a_{12}(v,y,z)-a_{23}(v,y,z)a_{12}(v,y,z)=a_{23}(v,y,z)L(y);$$

the latter can be rewritten in terms of (u,x,y) as

$$a_{22}(u,x,y)a_{13}(u,x,y)+a_{12}(u,x,y)-a_{23}(u,x,y)a_{12}(u,x,y)=a_{23}(u,x,y)L(x).$$

Replacing the $a_{13}(u,x,y)$ in this equality by (3.6), we get

$$a_{12}(u,x,y)=a_{22}(u,x,y)L(y)-L(x).\tag{3.7}$$

Inserting this into (3.3) and (3.6) we get respectively, by (3.4),

$$a_{34}(u,x,y)=L(y)-a_{33}(u,x,y)L(x)-a_{23}(u,x,y)L_1(y),$$

and

$$a_{13}(u,x,y)=a_{33}(u,x,y)L(x)+a_{23}(u,x,y)L(y)-L(y).\tag{3.8}$$

Also, from (3.4) and Lemma 3.1, we get

$$\exp(\alpha_3 u)L_1(x)M_3(x)=L_1(y)M_3(y).$$

This shows that if $\alpha_3\neq 0$ then the above $L_1(x)$ must be 0, and if $\alpha_3=0$ then $L_1(x)$ must ensure that $L_1(x)M_3(x)$ is a constant c. Gathering together these two cases, we can summarize that

$$L_1(x)=cM_3(x)^{-1},$$

where c is a complex constant satisfying $c\alpha_3=0$. Notice that this methodology will be used again in the following discussions. Hence we have

$$a_{34}(u,x,y)=L(y)-a_{33}(u,x,y)L(x)-ca_{23}(u,x,y)M_3(y)^{-1}.\tag{3.9}$$

Similarly, by (E3), we can write (E8) as

$$\begin{aligned}&a_{22}(v,y,z)a_{13}(u+v,x,z)\\&\quad=a_{13}(u,x,y)+a_{22}(v,y,z)a_{13}(v,y,z)+a_{23}(u,x,y)a_{12}(v,y,z).\end{aligned}\tag{3.10}$$

By (E5), we can write (E9) as

$$a_{13}(u+v,x,z)=a_{33}(v,y,z)a_{13}(u,x,y)+a_{13}(v,y,z)+a_{33}(v,y,z)a_{23}(u,x,y)a_{12}(v,y,z).$$

Using $a_{22}(u,x,y)a_{33}(u,x,y)=1$, this is clearly a consequence of (3.10). Now, using (3.7) and (3.8) for the substitution of a_{12} and a_{13} respectively, one can easily see that (3.10), so (E8) and (E9), hold. Also, by exactly the same arguments as in Lemma 2.1 of [15], we can derive from (E4) that

$$a_{23}(u,x,y)=\alpha u+N(x)-N(y),\tag{3.11}$$

where α is a complex constant, $N(x)$ is an arbitrary meromorphic function. The collection of the above proves (I) of Lemma 3.2.

Next, we turn to the other three cases where we always have

$$a_{44}(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y)\neq 1.$$

By (E3), (E4) and (E5), we can write (E6) as

$$\begin{aligned}&\big[a_{33}(u,x,y)a_{12}(u,x,y)-a_{34}(u,x,y)\big]a_{33}(v,y,z)a_{23}(v,y,z)\\&\quad=a_{23}(u,x,y)\big[a_{33}(v,y,z)a_{12}(v,y,z)-a_{34}(v,y,z)\big].\end{aligned}$$

This together with $a_{23}(u,x,y)\neq 0$, implies that there exists a meromorphic function $L_2(y)$ such that

$$a_{34}(u,x,y)=a_{33}(u,x,y)a_{12}(u,x,y)-a_{23}(u,x,y)L_2(y),$$

and

$$L_2(y)a_{33}(v,y,z)=L_2(z).$$

The latter in combination with Lemma 3.1 implies that there is a complex constant c_1 with $c_1\alpha_3=0$ such that

$$L_2(x)=c_1M_3(x)^{-1}.$$

Hence

$$a_{34}(u,x,y)=a_{33}(u,x,y)a_{12}(u,x,y)-c_1a_{23}(u,x,y)M_3(y)^{-1}.\tag{3.12}$$

Substituting by (E8), (E5) and (E3), then canceling and simplifying by (E1) and (E2), we can write (E9) as

$$\begin{aligned}&\big[1-a_{22}(u,x,y)a_{33}(u,x,y)\big]\big[a_{33}(v,y,z)a_{12}(v,y,z)-a_{34}(v,y,z)\big]\\&\quad=\big[a_{33}(u,x,y)a_{12}(u,x,y)-a_{34}(u,x,y)\big]\big[1-a_{22}(v,y,z)a_{33}(v,y,z)\big]a_{33}(v,y,z).\end{aligned}\tag{3.13}$$

Since $a_{22}(u,x,y)a_{33}(u,x,y)\neq 1$, this implies that there exists a meromorphic function $L_3(y)$ such that

$$a_{33}(u,x,y)a_{12}(u,x,y)-a_{34}(u,x,y)=\big[1-a_{22}(u,x,y)a_{33}(u,x,y)\big]L_3(y),$$

and

$$L_3(z)=a_{33}(v,y,z)L_3(y),$$

and the latter together with Lemma 3.1 implies that there is a complex constant c_2 with $c_2\alpha_3=0$ such that

$$L_3(x)=c_2M_3(x)^{-1}.$$

Hence

$$a_{33}(u,x,y)a_{12}(u,x,y)-a_{34}(u,x,y)=c_2\big[1-a_{22}(u,x,y)a_{33}(u,x,y)\big]M_3(y)^{-1}.\tag{3.14}$$

The combination of (3.12) and (3.14) gives

$$c_1a_{23}(u,x,y)=c_2\big[1-a_{22}(u,x,y)a_{33}(u,x,y)\big].\tag{3.15}$$

Again, by (E3), we can write (E8) as

$$\begin{aligned}a_{22}(v,y,z)a_{13}(u+v,x,z) = {} & a_{13}(u,x,y) + a_{22}(u+v,x,z)a_{33}(u,x,y)a_{13}(v,y,z) \\ & + \big[1 - a_{22}(v,y,z)a_{33}(v,y,z)\big]a_{33}(u,x,y)a_{12}(u,x,y) \\ & - \big[1 - a_{23}(u,x,y) - a_{22}(u,x,y)a_{33}(u,x,y)\big]a_{12}(v,y,z). \end{aligned} \tag{3.16}$$

This in combination with (E4) and (E3) enables us to write (E7) as

$$\begin{aligned}&\big[1 - a_{22}(v,y,z)a_{33}(v,y,z) - a_{23}(v,y,z)\big]\big[a_{13}(u,x,y) + a_{33}(u,x,y)a_{12}(u,x,y)\big] \\ &\quad = \big[1 - a_{22}(u,x,y)a_{33}(u,x,y) - a_{23}(u,x,y)\big] \\ &\qquad \times \big[a_{22}(v,y,z)a_{13}(v,y,z) + a_{12}(v,y,z) - a_{23}(v,y,z)a_{12}(v,y,z)\big]. \end{aligned} \tag{3.17}$$

Now we again consider two cases according to $1 - a_{22}(u,x,y)a_{33}(u,x,y) - a_{23}(u,x,y) = 0$ or not. When $1 - a_{22}(u,x,y)a_{33}(u,x,y) - a_{23}(u,x,y) = 0$, (3.17), so (E7), clearly holds, and (3.15) implies $c_1 = c_2$. Also (3.16) becomes

$$\begin{aligned}a_{22}(v,y,z)a_{13}(u+v,x,z) = {} & a_{13}(u,x,y) + a_{22}(u+v,x,z)a_{33}(u,x,y)a_{13}(v,y,z) \\ & + a_{23}(v,y,z)a_{33}(u,x,y)a_{12}(u,x,y). \end{aligned} \tag{3.18}$$

So in this case we can reformulate systems (B2) and (B3) in terms of (3.12), (3.18) and $a_{23}(u,x,y) = 1 - a_{22}(u,x,y)a_{33}(u,x,y)$. This proves (II) of Lemma 3.2.

When $1 - a_{22}(u,x,y)a_{33}(u,x,y) - a_{23}(u,x,y) \neq 0$, (3.17) implies that there exists a meromorphic function $L_4(y)$ such that

$$a_{13}(u,x,y) = \big[1 - a_{22}(u,x,y)a_{33}(u,x,y) - a_{23}(u,x,y)\big]L_4(y) - a_{33}(u,x,y)a_{12}(u,x,y),$$

and

$$a_{12}(u,x,y) = L_4(x) - a_{22}(u,x,y)L_4(y). \tag{3.19}$$

Substituting the latter into the former we get

$$a_{13}(u,x,y) = L_4(y) - a_{23}(u,x,y)L_4(y) - a_{33}(u,x,y)L_4(x). \tag{3.20}$$

Further, substituting a_{12} and a_{13} by (3.19) and (3.20) respectively, one can easily check that (3.16), so (E8), holds. Now, in view of (3.15), there exist two possibilities:

(i) If $c_1c_2 \neq 0$ (so $\alpha_3 = 0$), then by (3.12), (3.14) and (3.15), the solutions of systems (B2) and (B3) can be formulated by (3.19), (3.20) and $a_{23}(u,x,y) = c_3[1 - a_{22}(u,x,y)a_{33}(u,x,y)]$, $a_{34}(u,x,y) = a_{33}(u,x,y)a_{12}(u,x,y) - c_4a_{23}(u,x,y)M_3(y)^{-1}$, where $c_3 \neq 0, 1$ and $c_4 \neq 0$ are complex constants. This proves (III) of Lemma 3.2.

(ii) If $c_1 = c_2 = 0$, then by (3.12) and (3.14), the systems (B2) and (B3) can be reformulated by (3.19), (3.20) and $a_{34}(u,x,y) = a_{33}(u,x,y)a_{12}(u,x,y)$. This proves (IV) of Lemma 3.2, and the proof of Lemma 3.2 is complete. □

4. Proof of Theorem 1

In this section, we consider systems (B4) and (B5) under case (I) of Lemma 3.2 to prove Theorem 1. So we already have

$$\begin{cases} a_{22}(u,x,y)a_{33}(u,x,y) = 1, \\ a_{12}(u,x,y) = a_{22}(u,x,y)L(y) - L(x), \\ a_{23}(u,x,y) = \alpha u + N(x) - N(y) \neq 0, \\ a_{34}(u,x,y) = L(y) - a_{33}(u,x,y)L(x) - ca_{23}(u,x,y)M_3(y)^{-1}, \\ a_{13}(u,x,y) = a_{33}(u,x,y)L(x) + a_{23}(u,x,y)L(y) - L(y), \end{cases} \tag{4.1}$$

where α and c are complex constants such that $c\alpha_3 = 0$ with α_3 as in Lemma 1.1 and c is fixed throughout this section, $L(x)$ and $N(x)$ are meromorphic functions. We first consider system (B4) of function equations. By (E1)–(E3) and the first equality in (4.1), we can write (E10) as

$$\begin{aligned} a_{24}(u+v,x,z) = {} & a_{24}(u,x,y) + a_{22}(u,x,y)a_{24}(v,y,z) \\ & + a_{22}(u,x,y)a_{34}(u,x,y)a_{23}(v,y,z), \end{aligned} \tag{4.2}$$

and by (E1), (E2), (E5) and the first equality in (4.1), we see easily that (E11) is a consequence of (4.2). Substituting $a_{24}(u+v,x,z)$, $a_{23}(u+v,x,z)$ and a_{34} by (4.2), (E4) and the fourth equality in (4.1) respectively, then canceling and simplifying by (E1), (E2), the first equality in (4.1) and the second equality in Lemma 3.1, and noting $c\alpha_3 = 0$, we can rewrite (E12) as

$$\begin{aligned} & a_{22}(u,x,y)a_{23}(u,x,y)a_{24}(v,y,z) \\ & \quad = a_{23}(v,y,z)a_{24}(u,x,y) + a_{23}(u,x,y)a_{23}(v,y,z)L(x) - a_{23}(v,y,z)L(x) \\ & \qquad + a_{22}(u,x,y)a_{23}(v,y,z)L(y) + a_{22}(u,x,y)a_{23}(u,x,y)L(y) \\ & \qquad - a_{22}(u,x,y)a_{23}(u,x,y)a_{23}(v,y,z)L(y) - a_{22}(u+v,x,z)a_{23}(u,x,y)L(z) \\ & \qquad + cM_3(x)^{-1}a_{23}(u,x,y)^2a_{23}(v,y,z). \end{aligned} \tag{4.3}$$

Using (4.2), (4.3), the next-to-last and the last equality in (4.1) for the substitution of $a_{24}(u+v,x,z)$, $a_{24}(v,y,z)$, a_{34} and a_{13} respectively, then canceling and simplifying by (E1), (E2), (E4) and the first equality in (4.1), we get from (E15) that

$$\begin{aligned} a_{24}(u,x,y) = {} & L(x) - a_{23}(u,x,y)L(x) - a_{22}(u,x,y)L(y) \\ & + cM_3(x)^{-1}a_{23}(u,x,y) - cM_3(x)^{-1}a_{23}(u,x,y)^2. \end{aligned} \tag{4.4}$$

Using this to replace the $a_{24}(v)$ in (4.3), and using the first equality in (4.1) and the second equality in Lemma 3.1 to simplify, we get from (4.3) that

$$\begin{aligned} a_{24}(u,x,y) = {} & L(x) - a_{23}(u,x,y)L(x) - a_{22}(u,x,y)L(y) + cM_3(x)^{-1}a_{23}(u,x,y) \\ & - cM_3(x)^{-1}a_{23}(u,x,y)^2 - cM_3(x)^{-1}a_{23}(u,x,y)a_{23}(v,y,z). \end{aligned}$$

This together with (4.4) gives $-cM_3(x)^{-1}a_{23}(u,x,y)a_{23}(v,y,z) = 0$, which proves $c = 0$, and so by (4.4),

$$a_{24}(u,x,y) = L(x) - a_{23}(u,x,y)L(x) - a_{22}(u,x,y)L(y). \tag{4.5}$$

Next, using the second, the next-to-last, the last equality in (4.1) with $c = 0$, and (4.5) for the substitution of a_{12}, a_{34}, a_{13} and a_{24} respectively, one can check directly that (E10), (E13) and (E14) hold. Thus we can summarize that, if we only consider the first four categories (B1)–(B4), then

$$\begin{cases} a_{12}(u,x,y) = a_{22}(u,x,y)L(y) - L(x), \\ a_{23}(u,x,y) = \alpha u + N(x) - N(y), \\ a_{34}(u,x,y) = L(y) - a_{33}(u,x,y)L(x), \\ a_{13}(u,x,y) = a_{33}(u,x,y)L(x) + a_{23}(u,x,y)L(y) - L(y), \\ a_{24}(u,x,y) = L(x) - a_{23}(u,x,y)L(x) - a_{22}(u,x,y)L(y), \end{cases} \tag{4.6}$$

where $L(x)$ and $N(x)$ are meromorphic functions, α is a complex constant.

Next, we consider system (B5) of function equations to give a general solution for $a_{14}(u,x,y)$. We first note that (E17) implies (E19) by (E3), (E5) and the first equality in (4.1). Next, by (E17) and the first equality in (4.1), we can rewrite (E21) as

$$\begin{aligned} & a_{23}(v,y,z)a_{23}(u+v,x,z)a_{14}(u,x,y) \\ & \quad = -a_{22}(v,y,z)a_{24}(u,x,y)a_{13}(u+v,x,z) + a_{22}(u+v,x,z)a_{34}(v,y,z)a_{13}(u,x,y) \\ & \qquad - a_{22}(u+v,x,z)a_{34}(u,x,y)a_{13}(v,y,z) + a_{13}(u,x,y)a_{24}(u+v,x,z) \\ & \qquad - a_{23}(v,y,z)a_{34}(u,x,y)a_{24}(u+v,x,z) - a_{12}(v,y,z)a_{24}(u,x,y) \end{aligned}$$

$$
\begin{aligned}
&+ a_{12}(u,x,y)a_{24}(v,y,z) - a_{23}(u+v,x,z)a_{24}(u,x,y)a_{24}(v,y,z) \\
&- a_{34}(u,x,y)a_{12}(u+v,x,z) + a_{22}(v,y,z)a_{12}(u,x,y)a_{34}(u+v,x,z).
\end{aligned} \tag{4.7}
$$

Substituting a_{12}, a_{34}, a_{13} and a_{24} by (4.6), then canceling and simplifying by (E4) and the first equality in (4.1), we get from (4.7),

$$a_{14}(u,x,y) = 2L(x)L(y) - a_{23}(u,x,y)L(x)L(y) - a_{33}(u,x,y)L(x)L(x) - a_{22}(u,x,y)L(y)L(y). \tag{4.8}$$

Also, using (4.6) and (4.8) for the substitution of a_{12}, a_{34}, a_{13}, a_{24} and a_{14}, direct computations show that (E16), (E18), (E20) and (E22) hold. The combination of (4.6) and (4.8) proves Theorem 1.

5. Proof of Theorem 2

In this section, we proceed our investigations under case (II) of Lemma 3.2 to prove Theorem 2, so we have had

$$
\begin{cases}
a_{44}(u,x,y) = a_{22}(u,x,y)a_{33}(u,x,y) \neq 1, \\
a_{23}(u,x,y) = 1 - a_{22}(u,x,y)a_{33}(u,x,y), \\
a_{34}(u,x,y) = a_{33}(u,x,y)a_{12}(u,x,y) - ca_{23}(u,x,y)M_3(y)^{-1},
\end{cases} \tag{5.1}
$$

where c is a complex constant, which is fixed throughout this section, such that $c\alpha_3 = 0$ with α_3 as in Lemma 1.1. Thus by (E1), (E2), we can write (E10) as

$$
\begin{aligned}
&a_{24}(u+v,x,z) \\
&\quad = a_{24}(u,x,y) + a_{22}(u,x,y)^2 a_{33}(u,x,y)a_{24}(v,y,z) \\
&\qquad + a_{22}(u,x,y)a_{34}(u,x,y)a_{23}(v,y,z) + a_{22}(u,x,y)a_{33}(u,x,y)a_{12}(u,x,y)a_{23}(v,y,z) \\
&\qquad - a_{22}(u,x,y)a_{23}(u,x,y)a_{12}(v,y,z).
\end{aligned} \tag{5.2}
$$

Substituting $a_{24}(u+v,x,z)$ and a_{34} by (5.2) and the last equality in (5.1) respectively, then using the second equality in (5.1) and the last equality in Lemma 3.1, one can check that (E11) holds. Using (5.2), (E4) and the last equality in (5.1) for the substitution of $a_{24}(u+v,x,z)$, $a_{23}(u+v,x,z)$ and $a_{34}(v)$ respectively, then using the second equality in (5.1) and the second equality in Lemma 3.1 for simplification, we can rewrite (E12) as

$$
\begin{aligned}
&2a_{22}(u,x,y)^2 a_{33}(u,x,y)a_{23}(u,x,y)\big[a_{24}(v,y,z) + a_{12}(v,y,z)\big] \\
&\quad = \big[2a_{22}(u,x,y)^2 a_{33}(u,x,y)a_{34}(u,x,y) + 2a_{22}(u,x,y)^2 a_{33}(u,x,y)^2 a_{12}(u,x,y) \\
&\qquad + 2a_{22}(u,x,y)a_{33}(u,x,y)a_{24}(u,x,y) - a_{22}(u,x,y)a_{34}(u,x,y) \\
&\qquad - a_{22}(u,x,y)a_{33}(u,x,y)a_{12}(u,x,y) + ca_{22}(u,x,y)a_{23}(u,x,y)M_3(y)^{-1}\big]a_{23}(v,y,z).
\end{aligned}
$$

This implies that there exists a meromorphic function $L_5(y)$ such that

$$a_{24}(u,x,y) = a_{23}(u,x,y)L_5(x) - a_{12}(u,x,y),$$

and

$$
\begin{aligned}
&2a_{22}(u,x,y)^2 a_{33}(u,x,y)a_{34}(u,x,y) + 2a_{22}(u,x,y)^2 a_{33}(u,x,y)^2 a_{12}(u,x,y) \\
&\qquad + 2a_{22}(u,x,y)a_{33}(u,x,y)a_{24}(u,x,y) - a_{22}(u,x,y)a_{34}(u,x,y) \\
&\qquad - a_{22}(u,x,y)a_{33}(u,x,y)a_{12}(u,x,y) + ca_{22}(u,x,y)a_{23}(u,x,y)M_3(y)^{-1} \\
&\quad = 2a_{22}(u,x,y)^2 a_{33}(u,x,y)a_{23}(u,x,y)L_5(y);
\end{aligned}
$$

and using the former and the last equality in (5.1) for the substitution of $a_{24}(u,x,y)$ and $a_{34}(u,x,y)$ respectively in the latter, and simplifying by Lemma 3.1 and the second equality in (5.1), we get

$$a_{12}(u,x,y) = L(x) - a_{22}(u,x,y)L(y), \tag{5.3}$$

where $2L(x) = L_5(x) + cM_3(x)^{-1}$. So the former gives

$$a_{24}(u,x,y) = 2a_{23}(u,x,y)L(x) - L(x) + a_{22}(u,x,y)L(y) - ca_{23}(u,x,y)M_3(x)^{-1}. \tag{5.4}$$

Now, direct computations show that (5.2), so (E10), holds for the solutions given by (5.3) and (5.4). For the general solution of $a_{13}(u,x,y)$, we consider (E8) and (E13)–(E15). By (E3), the second equality in (5.1) and the first equality in Lemma 3.1, we can write (E8) as

$$\begin{aligned} a_{13}(u+v,x,z) &= a_{13}(u,x,y)a_{22}(-v,z,y) + a_{22}(u,x,y)a_{33}(u,x,y)a_{13}(v,y,z) \\ &\quad + a_{33}(u,x,y)a_{12}(u,x,y)a_{22}(-v,z,y)a_{23}(v,y,z). \end{aligned} \tag{5.5}$$

Inserting (5.2), (5.5) and (E4) into (E13), then canceling and simplifying by (E1), (E2) and (5.1), we get

$$\begin{aligned} &a_{22}(u,x,y)a_{23}(u,x,y) \\ &\quad \times \big[a_{23}(v,y,z)a_{12}(v,y,z) - a_{22}(v,y,z)a_{13}(v,y,z) + a_{24}(v,y,z) + cM_3(y)^{-1}a_{23}(v,y,z)^2\big] \\ &\quad = a_{23}(v,y,z)\big[a_{24}(u,x,y) - a_{22}(u,x,y)a_{13}(u,x,y) - a_{12}(u,x,y)a_{23}(u,x,y)\big]. \end{aligned}$$

This implies that there exists a meromorphic function $L_6(y)$ such that

$$\begin{aligned} a_{22}(u,x,y)a_{13}(u,x,y) &= a_{24}(u,x,y) - a_{12}(u,x,y)a_{23}(u,x,y) \\ &\quad - a_{22}(u,x,y)a_{23}(u,x,y)\big[2L_6(y) + cM_3(y)^{-1}\big], \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &a_{23}(u,x,y)a_{12}(u,x,y) - a_{22}(u,x,y)a_{13}(u,x,y) + a_{24}(u,x,y) + cM_3(x)^{-1}a_{23}(u,x,y)^2 \\ &\quad = a_{23}(u,x,y)\big[2L_6(x) + cM_3(x)^{-1}\big]. \end{aligned}$$

The latter together with the former and the second equality in (5.1) gives

$$a_{12}(u,x,y) = L_6(x) - a_{22}(u,x,y)L_6(y).$$

This in combination with (5.3) yields

$$\big[L(x) - L_6(x)\big] = a_{22}(u,x,y)\big[L(y) - L_6(y)\big].$$

This together with the first equality in Lemma 3.1 implies further that

$$L_6(x) = L(x) + c_5M_2(x),$$

where c_5 is a complex constant satisfying $c_5\alpha_2 = 0$. So we get from (5.6),

$$\begin{aligned} a_{22}(u,x,y)a_{13}(u,x,y) &= a_{24}(u,x,y) - a_{23}(u,x,y)a_{12}(u,x,y) \\ &\quad - a_{22}(u,x,y)a_{23}(u,x,y)\big[2L(y) + 2c_5M_2(y) + cM_3(y)^{-1}\big]. \end{aligned}$$

Using (5.3), (5.4), Lemma 3.1 and the second equality in (5.1), this gives

$$\begin{aligned} a_{13}(u,x,y) &= a_{22}(u,x,y)a_{33}(u,x,y)L(y) - a_{33}(u,x,y)L(x) - 2c_5M_2(y)a_{23}(u,x,y) \\ &\quad + cM_3(y)^{-1}a_{22}(u,x,y)a_{33}(u,x,y) - cM_3(x)^{-1}a_{22}(-u,y,x). \end{aligned} \tag{5.7}$$

Using (5.4), (5.7) and the last equality in (5.1) for the substitutions of a_{24}, a_{13} and a_{34} respectively, then (5.3) for the substitution of a_{12}, and then simplifying by (E1), (E2), Lemma 3.1 and the second equality in (5.1), (E14) yields

$$ca_{33}(u+v,x,z)a_{23}(u,x,y)a_{23}(v,y,z)a_{23}(u+v,x,z) = 0,$$

this proves that $c = 0$. So by (5.4) and (5.7) we get

$$\begin{cases} a_{24}(u,x,y) = 2a_{23}(u,x,y)L(x) - L(x) + a_{22}(u,x,y)L(y), \\ a_{13}(u,x,y) = a_{22}(u,x,y)a_{33}(u,x,y)L(y) - a_{33}(u,x,y)L(x) - 2c_5M_2(y)a_{23}(u,x,y). \end{cases} \tag{5.8}$$

Using these for the substitution of a_{24} and a_{13}, then simplifying by (E1), (E2) and the second equality in (5.1), (E15) yields

$$c_5a_{23}(u,x,y)a_{23}(v,y,z)a_{23}(u+v,x,z) = 0,$$

which implies that $c_5 = 0$. Therefore, from (5.1), (5.3) and (5.8) with $c_5 = 0$, we obtain

$$\begin{cases} a_{12}(u,x,y) = L(x) - a_{22}(u,x,y)L(y), \\ a_{23}(u,x,y) = 1 - a_{22}(u,x,y)a_{33}(u,x,y), \\ a_{34}(u,x,y) = a_{33}(u,x,y)a_{12}(u,x,y), \\ a_{13}(u,x,y) = -a_{34}(u,x,y), \\ a_{24}(u,x,y) = 2a_{23}(u,x,y)L(x) - a_{12}(u,x,y), \end{cases} \tag{5.9}$$

where $L(x)$ is an arbitrary meromorphic function.

The remainder of this section is devoted to the consideration of equation system (B5) to give the general solution of $a_{14}(u,x,y)$ under case (II) of Lemma 3.2. Using (E16) for the substitution of $a_{14}(u+v,x,z)$ in (E17), and using (E3), (E4) and (5.9) for simplification, we get

$$\begin{aligned} &a_{23}(v,y,z)\bigl[a_{14}(u,x,y) + a_{33}(u,x,y)a_{12}(u,x,y)^2\bigr] \\ &\quad = a_{22}(u,x,y)a_{23}(u,x,y)\bigl[a_{22}(v,y,z)a_{14}(v,y,z) \\ &\qquad + a_{12}(v,y,z)a_{12}(v,y,z) - 2a_{23}(v,y,z)a_{12}(v,y,z)L(y)\bigr]. \end{aligned}$$

This implies that there exists a meromorphic function $L_7(y)$ such that

$$a_{14}(u,x,y) = a_{22}(u,x,y)a_{23}(u,x,y)L_7(y) - a_{33}(u,x,y)a_{12}(u,x,y)^2,$$

and

$$a_{23}(v,y,z)L_7(y) = a_{22}(v,y,z)a_{14}(v,y,z) + a_{12}(v,y,z)^2 - 2a_{23}(v,y,z)a_{12}(v,y,z)L(y).$$

Using the former for the substitution of $a_{14}(v,y,z)$ in the latter, we get

$$L_7(y) + L(y)^2 = a_{22}(v,y,z)^2\bigl[L_7(z) + L(z)^2\bigr],$$

which, together with Lemma 3.1, implies that

$$L_7(y) = c_6^2 M_2(y)^2 - L(y)^2,$$

where c_6 is a complex constant satisfying $c_6\alpha_2 = 0$. So, by (5.9) and Lemma 3.1, we can write

$$\begin{aligned} a_{14}(u,x,y) = {} & c_6^2 M_2(x)M_2(y)a_{23}(u,x,y) - a_{33}(u,x,y)L(x)^2 \\ & - a_{22}(u,x,y)L(y)^2 + 2a_{22}(u,x,y)a_{33}(u,x,y)L(x)L(y). \end{aligned} \tag{5.10}$$

Now, (E16) clearly holds to the solutions given by (5.9) and (5.10). And using (5.9) and (5.10) for the substitutions of a_{24}, a_{34}, a_{12} and a_{14}, then using (E4) and the second equality in (5.9) to simplify, we can write (E18) as

$$\begin{aligned} &c_6^2 a_{22}(v,y,z)a_{33}(v,y,z)a_{23}(u,x,y)a_{23}(u+v,x,z)M_2(x)M_2(z) \\ &\quad = c_6^2 a_{33}(v,y,z)a_{23}(u+v,x,z)a_{23}(u,x,y)M_2(x)M_2(y), \end{aligned}$$

which is true by virtue of Lemma 3.1, so proves the validity of (E18). Also, similar arguments can show the validity of (E19) to (E22). Therefore, the combination of (5.9) and (5.10), together with $a_{22}(u,x,y)a_{33}(u,x,y) \neq 1$, proves Theorem 2.

6. The systems (B4) and (B5) under case (III)

In this section, we continue our investigation under case (III) of Lemma 3.2. We will prove that, when combined with the equations from categories (B4) and (B5), case (III) of Lemma 3.2 always leads to contradictions; so this case has no contribution to the solution of the QCYBE. Our conditions are

$$\begin{cases} a_{44}(u,x,y) = a_{22}(u,x,y)a_{33}(u,x,y) \neq 1, \\ a_{12}(u,x,y) = L(x) - a_{22}(u,x,y)L(y), \\ a_{23}(u,x,y) = c'\bigl[1 - a_{22}(u,x,y)a_{33}(u,x,y)\bigr], \quad c' \neq 1, \\ a_{34}(u,x,y) = a_{33}(u,x,y)a_{12}(u,x,y) - ca_{23}(u,x,y)M_3(y)^{-1}, \\ a_{13}(u,x,y) = L(y) - a_{23}(u,x,y)L(y) - a_{33}(u,x,y)L(x), \end{cases} \tag{6.1}$$

where the α_3 in Lemma 1.1 is $\alpha_3 = 0$, c and c' are two nonzero complex constants, and $L(x)$ is a meromorphic function. We shall separate our discussion into two cases according to $c' = -1$ or not. We first consider the case $c' = -1$, so the third equality in (6.1) gives

$$a_{23}(u, x, y) = a_{22}(u, x, y)a_{33}(u, x, y) - 1. \tag{6.2}$$

By (6.1), (6.2) and Lemma 3.1, we can write (E12) as

$$\begin{aligned} &a_{24}(u + v, x, z) \\ &\quad = a_{24}(u, x, y) + a_{22}(u, x, y)^2 a_{33}(u, x, y)a_{24}(v, y, z) + a_{22}(u, x, y)^2 a_{33}(u, x, y)L(y) \\ &\qquad - a_{22}(u, x, y)L(y) + a_{22}(u + v, x, z)L(z) - a_{22}(u, x, y)a_{22}(u + v, x, z)a_{33}(u, x, y)L(z) \\ &\qquad - cM_3(z)^{-1}a_{22}(u + v, x, z)a_{23}(u, x, y) + cM_3(z)^{-1}a_{22}(u, x, y)a_{33}(v, y, z)^{-1}a_{23}(u, x, y). \end{aligned} \tag{6.3}$$

Substituting a_{12}, a_{13} and $a_{24}(u + v, x, z)$ by (6.1) and (6.3), then canceling and simplifying by (E4) and (6.2), we can write (E13) as

$$\begin{aligned} &\big[a_{24}(u, x, y) + L(x) - a_{22}(u, x, y)L(y)\big]a_{33}(v, y, z)a_{23}(v, y, z) \\ &\quad = a_{22}(u, x, y)a_{23}(u, x, y)\big[a_{33}(v, y, z)L(y) \\ &\qquad + a_{33}(v, y, z)a_{24}(v, y, z) - a_{22}(v, y, z)a_{33}(v, y, z)L(z) + cM_3(z)^{-1}a_{23}(v, y, z)^2\big]. \end{aligned}$$

Thus there exists a meromorphic function $L_8(y)$ such that

$$a_{24}(u, x, y) = a_{22}(u, x, y)L(y) - L(x) + a_{22}(u, x, y)a_{23}(u, x, y)L_8(y)$$

and

$$\begin{aligned} &a_{33}(v, y, z)L(y) + a_{33}(v, y, z)a_{24}(v, y, z) - a_{22}(v, y, z)a_{33}(v, y, z)L(z) + cM_3(z)^{-1}a_{23}(v, y, z)^2 \\ &\quad = a_{33}(v, y, z)a_{23}(v, y, z)L_8(y). \end{aligned}$$

Using the former for the substitution of $a_{24}(v, y, z)$ in the latter, and using the second equality in Lemma 3.1 for simplification, we get

$$L_8(y) + cM_3(y)^{-1} = a_{22}(v, y, z)\big[L_8(z) + cM_3(z)^{-1}\big].$$

This implies that

$$L_8(y) = c_7 M_2(y) - cM_3(y)^{-1},$$

where c_7 is a complex constant satisfying $c_7\alpha_2 = 0$. Hence

$$\begin{aligned} a_{24}(u, x, y) = {} & a_{22}(u, x, y)L(y) - L(x) + c_7 M_2(x)a_{23}(u, x, y) \\ & - cM_3(y)^{-1}a_{22}(u, x, y)a_{23}(u, x, y). \end{aligned} \tag{6.4}$$

Substituting a_{13} and a_{24} by (6.1) and (6.4), then canceling and simplifying by (E4), (6.2) and Lemma 3.1, we can write (E15) as

$$a_{22}(v, y, z)a_{23}(u, x, y)a_{23}(v, y, z)a_{23}(u + v, x, z)\big[c_7 M_2(z)M_3(z) - c\big] = 0,$$

i.e.

$$c_7 M_2(z)M_3(z) - c = 0.$$

Also, from $c \neq 0$ we see that $c_7 \neq 0$ (so $\alpha_2 = 0$), and whence

$$M_2(z)M_3(z) = c/c_7,$$

which together with Lemma 3.1 yields

$$a_{22}(u, x, y)a_{33}(u, x, y) = \frac{M_2(x)}{M_2(y)}\frac{M_3(x)}{M_3(y)} = 1,$$

i.e. $a_{23}(u,x,y)=0$ by (6.2). This contradicts to our assumption (1.2), and thus shows that (E15), so the Yang–Baxter equation, has no solution of the form given by (6.1) and (6.4).

Next, we consider the case $c' \neq -1$, hence the third equality in (6.1) becomes

$$a_{23}(u,x,y)=c'\big[1-a_{22}(u,x,y)a_{33}(u,x,y)\big] \quad \text{with } c' \neq \pm 1. \tag{6.5}$$

Substituting (E10) into (E12), then simplifying by (E3), (E4), (6.5) and the fourth equality in (6.1), we get on noting $c' \neq -1$,

$$\begin{aligned}&\big[a_{33}(u,x,y)a_{24}(u,x,y)+a_{22}(u,x,y)a_{33}(u,x,y)^2a_{12}(u,x,y)\\&\qquad -a_{33}(u,x,y)a_{23}(u,x,y)a_{12}(u,x,y)+ca_{23}(u,x,y)^2M_3(y)^{-1}\big]a_{23}(v,y,z)\\&\quad =a_{22}(u,x,y)a_{33}(u,x,y)a_{23}(u,x,y)\big[a_{24}(v,y,z)+a_{12}(v,y,z)\big].\end{aligned}$$

This implies that there exists a meromorphic function $L_9(y)$ such that

$$a_{24}(u,x,y)=a_{23}(u,x,y)L_9(x)-a_{12}(u,x,y)$$

and

$$\begin{aligned}&a_{33}(u,x,y)a_{24}(u,x,y)+a_{22}(u,x,y)a_{33}(u,x,y)^2a_{12}(u,x,y)\\&\qquad -a_{33}(u,x,y)a_{23}(u,x,y)a_{12}(u,x,y)+ca_{23}(u,x,y)^2M_3(y)^{-1}\\&\quad =a_{22}(u,x,y)a_{33}(u,x,y)a_{23}(u,x,y)L_9(y).\end{aligned}$$

Substituting the former into the latter, and simplifying by (6.5) and the second equality in (6.1), we get

$$L_{10}(x)=a_{22}(u,x,y)L_{10}(y),$$

where $L_{10}(x)=L_9(x)-(1+c')c'^{-1}L(x)+cc'M_3(x)^{-1}$. This together with Lemma 3.1 gives

$$L_{10}(x)=c_8M_2(x),$$

where c_8 is a complex constant satisfying $c_8\alpha_2=0$. And thus

$$\begin{aligned}a_{24}(u,x,y)={}&c'L(x)-(1+c')a_{22}(u,x,y)a_{33}(u,x,y)L(x)+a_{22}(u,x,y)L(y)\\&+c_8M_2(x)a_{23}(u,x,y)-cc'M_3(x)^{-1}a_{23}(u,x,y).\end{aligned} \tag{6.6}$$

Using (6.1) and (6.6) for the substitutions of a_{12}, a_{13} and a_{24}, then canceling and simplifying by (6.5), (E4) and $cc' \neq 0$, we can write (E13) as

$$M_3(x)^{-1}a_{23}(u+v,x,z)a_{23}(v,y,z)-M_3(y)^{-1}a_{23}(u+v,x,z)a_{23}(v,y,z)a_{22}(u,x,y)=0.$$

This together with Lemma 3.1 and (1.2) yields

$$1-a_{22}(u,x,y)a_{33}(u,x,y)=0,$$

which in combination with (6.5) implies $a_{23}(u,x,y)=0$, a contradiction of (1.2). This shows that, in the case of $c' \neq -1$, (E13), so the Yang–Baxter equation, does not have any solution either as in the first case.

7. Proof of Theorem 3

In this last section, we continue our investigations under case (IV) of Lemma 3.2 to prove Theorem 3, and accomplish this paper. We now have had

$$\begin{cases}a_{44}(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y)\neq 1,\\ a_{23}(u,x,y)\neq 1-a_{22}(u,x,y)a_{33}(u,x,y),\\ a_{12}(u,x,y)=L(x)-a_{22}(u,x,y)L(y),\\ a_{34}(u,x,y)=a_{33}(u,x,y)a_{12}(u,x,y),\\ a_{13}(u,x,y)=L(y)-a_{23}(u,x,y)L(y)-a_{33}(u,x,y)L(x),\end{cases} \tag{7.1}$$

where $L(x)$ is an arbitrary meromorphic function. We first consider system (B4) of function equations under the conditions of (7.1). We note at first that (E11) is a consequence of (E10) under (7.1). Next, inserting (E10) into (E13), using (7.1) for the substitutions of a_{12}, a_{34} and a_{13}, then using (E4) for cancelation and simplification, we get

$$\begin{aligned}&a_{22}(u,x,y)a_{23}(u,x,y)\big[a_{24}(v,y,z)-a_{22}(v,y,z)L(z)+a_{22}(v,y,z)a_{33}(v,y,z)L(y)\big]\\&\quad=\big[a_{24}(u,x,y)+a_{22}(u,x,y)a_{33}(u,x,y)L(x)-a_{23}(u,x,y)L(x)\\&\qquad+a_{22}(u,x,y)a_{23}(u,x,y)L(y)-a_{22}(u,x,y)L(y)\big]a_{23}(v,y,z).\end{aligned}$$

This together with $a_{23}(u,x,y)\neq 0$ implies that there exists a meromorphic function $L_{11}(y)$ such that

$$a_{24}(u,x,y)=a_{22}(u,x,y)L(y)-a_{22}(u,x,y)a_{33}(u,x,y)L(x)+a_{23}(u,x,y)L_{11}(x)$$

and

$$\begin{aligned}&a_{24}(u,x,y)+a_{22}(u,x,y)a_{33}(u,x,y)L(x)-a_{23}(u,x,y)L(x)+a_{22}(u,x,y)a_{23}(u,x,y)L(y)\\&\quad-a_{22}(u,x,y)L(y)=a_{22}(u,x,y)a_{23}(u,x,y)L_{11}(y).\end{aligned}$$

Inserting the former into the latter, we get

$$a_{22}(u,x,y)\big[L_{11}(y)-L(y)\big]=L_{11}(x)-L(x),$$

which in combination with Lemma 3.1 implies that

$$L_{11}(x)=L(x)+c_9M_2(x),$$

where c_9 is a complex constant satisfying $c_9\alpha_2=0$. And therefore we get

$$\begin{aligned}a_{24}(u,x,y)&=a_{22}(u,x,y)L(y)+a_{23}(u,x,y)L(x)\\&\qquad-a_{22}(u,x,y)a_{33}(u,x,y)L(x)+c_9M_2(x)a_{23}(u,x,y).\end{aligned}\tag{7.2}$$

Using (7.1) and (7.2) for the substitutions of a_{13} and a_{24}, then using (E4) and (1.2) for cancelation and simplification, we can write (E15) as

$$c_9M_2(x)a_{33}(u,x,y)-c_9M_2(y)+c_9M_2(y)a_{23}(u,x,y)=0,$$

i.e., by Lemma 3.1,

$$c_9\big[a_{23}(u,x,y)+a_{22}(u,x,y)a_{33}(u,x,y)-1\big]=0.$$

This together with the second inequality in (7.1) gives $c_9=0$, so by (7.2) we get

$$a_{24}(u,x,y)=a_{22}(u,x,y)L(y)+a_{23}(u,x,y)L(x)-a_{22}(u,x,y)a_{33}(u,x,y)L(x).\tag{7.3}$$

Now, using (7.1) and (7.3) for the substitutions of a_{34}, a_{13} and a_{24}, then using (E4) for cancelation and simplification, we can easily check the validity of (E12) and (E14).

Next, we turn to consider system (B5) of function equations under case (IV). Let

$$\begin{aligned}f(u,x,y)&=a_{14}(u,x,y)-a_{22}(u,x,y)a_{33}(u,x,y)L(x)L(y)-L(x)L(y)\\&\qquad+a_{23}(u,x,y)L(x)L(y)+a_{33}(u,x,y)L(x)^2+a_{22}(u,x,y)L(y)^2.\end{aligned}\tag{7.4}$$

Using (E16) and (E17) for the cancelation of $a_{14}(u+v,x,z)$, then using (7.1) and (7.3) for the substitutions of a_{34}, a_{13}, a_{12} and a_{24}, and using (E4) for cancelation and simplification, we get

$$a_{22}(u,x,y)a_{23}(u,x,y)a_{22}(v,y,z)f(v,y,z)=a_{23}(v,y,z)f(u,x,y).$$

This implies that there exists a meromorphic function $L_{12}(y)$ such that

$$a_{22}(v,y,z)a_{23}(v,y,z)^{-1}f(v,y,z)=a_{22}(u,x,y)^{-1}a_{23}(u,x,y)^{-1}f(u,x,y)=L_{12}(y),$$

so

$$f(u,x,y)=a_{22}(u,x,y)a_{23}(u,x,y)L_{12}(y),$$

and

$$L_{12}(x)=a_{22}(u,x,y)^2L_{12}(y).$$

The latter together with Lemma 3.1 implies that

$$L_{12}(x)=c_{10}M_2(x)^2,$$

where c_{10} is a complex constant satisfying $c_{10}\alpha_2=0$, and whence, by Lemma 3.1,

$$f(u,x,y)=c_{10}a_{23}(u,x,y)M_2(x)M_2(y). \tag{7.5}$$

The combination of (7.4) and (7.5) gives

$$\begin{aligned}a_{14}(u,x,y)={}&c_{10}a_{23}(u,x,y)M_2(x)M_2(y)-a_{33}(u,x,y)L(x)^2-a_{23}(u,x,y)L(x)L(y)+L(x)L(y)\\&+a_{22}(u,x,y)a_{33}(u,x,y)L(x)L(y)-a_{22}(u,x,y)L(y)^2.\end{aligned} \tag{7.6}$$

Now, using (7.1), (7.3) and (7.6) for the substitutions of a_{12}, a_{34}, a_{13}, a_{24} and a_{14}, then using (E4) and Lemma 3.1 for cancelation and simplification, one can check directly that (E18) and (E19) hold. Again, using (7.1), (7.3) and (7.4) for the substitutions of a_{12}, a_{34}, a_{13}, a_{24} and a_{14}, then using (E4) for cancelation and simplification, (E20) can be written as

$$\begin{aligned}&a_{33}(u,x,y)f(u+v,x,z)\\&\quad=a_{22}(v,y,z)a_{33}(u,x,y)a_{33}(v,y,z)^2f(u,x,y)-a_{23}(u,x,y)a_{23}(u,x,y)f(v,y,z)\\&\qquad-a_{22}(u,x,y)a_{33}(u,x,y)a_{23}(u,x,y)a_{23}(v,y,z)f(v,y,z)+f(v,y,z),\end{aligned} \tag{7.7}$$

and using (E17) for the substitution of $a_{14}(u+v,x,z)$, then using (7.1), (7.3) and (7.4) for the substitutions of a_{12}, a_{34}, a_{13}, a_{24} and a_{14}, and then using (E4) for cancelation and simplification, (E21) can be written as

$$\begin{aligned}&a_{23}(v,y,z)f(u,x,y)\big[1-a_{22}(u,x,y)^2a_{33}(u,x,y)^2-a_{23}(u,x,y)^2\big]\\&\quad=a_{22}(u,x,y)a_{33}(u,x,y)a_{23}(u,x,y)f(u,x,y)\big[1-a_{22}(v,y,z)^2a_{33}(v,y,z)^2+a_{23}(v,y,z)^2\big].\end{aligned} \tag{7.8}$$

By (7.5) and in view of $a_{23}\neq 0$, both (7.7) and (7.8) are reduced to

$$\begin{aligned}&c_{10}a_{23}(v,y,z)\big[1-a_{22}(u,x,y)^2a_{33}(u,x,y)^2-a_{23}(u,x,y)^2\big]\\&\quad=c_{10}a_{22}(u,x,y)a_{33}(u,x,y)a_{23}(u,x,y)\big[1-a_{22}(v,y,z)^2a_{33}(v,y,z)^2+a_{23}(v,y,z)^2\big].\end{aligned} \tag{7.9}$$

Further, using (7.1) and (7.3) for the substitutions of a_{12}, a_{34}, a_{13}, a_{24}, but with the coefficients of a_{14} unchanged, then using (7.4) for the substitution of a_{14}, and then using (E4) for cancelation and simplification, (E22) can be written as

$$\begin{aligned}&a_{22}(v,y,z)a_{33}(v,y,z)a_{13}(u,x,y)f(u+v,x,z)-a_{13}(u+v,x,z)f(u,x,y)\\&\quad-a_{23}(u+v,x,z)a_{13}(v,y,z)f(u,x,y)-a_{23}(u+v,x,z)a_{24}(u,x,y)f(v,y,z)\\&\quad+a_{24}(v,y,z)f(u+v,x,z)-a_{22}(u,x,y)a_{33}(u,x,y)a_{24}(u+v,x,z)f(v,y,z)\\&\quad-a_{33}(u,x,y)a_{12}(u,x,y)f(u+v,x,z)-a_{22}(u,x,y)a_{33}(u,x,y)a_{12}(v,y,z)f(u+v,x,z)\\&\quad+a_{12}(u+v,x,z)f(v,y,z)+a_{22}(v,y,z)a_{33}(v,y,z)a_{33}(u+v,x,z)a_{12}(u+v,x,z)f(u,x,y)=0.\end{aligned}$$

Again, using (7.1) and (7.3) for the substitutions of a_{13}, a_{24} and a_{12}, and using (E4), (7.5) and Lemma 3.1 for simplification, this is reduced to

$$\begin{aligned}&c_{10}\big[a_{22}(u,x,y)a_{22}(v,y,z)L(z)-L(x)\big]\big[1-a_{23}(u,x,y)^2-a_{22}(u,x,y)^2a_{33}(u,x,y)^2\big]a_{23}(v,y,z)\\&\quad=c_{10}\big[a_{22}(u,x,y)a_{22}(v,y,z)L(z)-L(x)\big]\big[1+a_{23}(v,y,z)^2-a_{22}(v,y,z)^2a_{33}(v,y,z)^2\big]\\&\qquad\times a_{22}(u,x,y)a_{33}(u,x,y)a_{23}(u,x,y),\end{aligned}$$

which is clearly true by virtue of (7.9), and so proves the validity of (E22) under (7.9).

Thus, we now consider two cases according to the c_{10} in (7.9) being 0 or not to continue our investigation. If $c_{10}=0$, then (7.9), and so (7.7) and (7.8), i.e., (E20) and (E21), hold. If $c_{10}\neq 0$, by (7.9) we see that there exists a meromorphic function $L_{13}(x)$ such that

$$1-a_{22}(u,x,y)^2a_{33}(u,x,y)^2=2a_{23}(u,x,y)L_{13}(x)-a_{23}(u,x,y)^2, \tag{7.10}$$

and

$$1-a_{22}(u,x,y)^2a_{33}(u,x,y)^2-a_{23}(u,x,y)^2=a_{22}(u,x,y)a_{33}(u,x,y)a_{23}(u,x,y)L_{13}(y).$$

Substituting the former into the latter, we get

$$a_{23}(u,x,y)=L_{13}(x)-a_{22}(u,x,y)a_{33}(u,x,y)L_{13}(y). \tag{7.11}$$

By (7.10) and (7.11), one can derive easily that

$$1-L_{13}(x)^2=a_{22}(u,x,y)^2a_{33}(u,x,y)^2\bigl[1-L_{13}(y)^2\bigr].$$

This together with Lemma 3.1 proves

$$1-L_{13}(x)^2=c_{11}M_2(x)^2M_3(x)^2, \quad \text{i.e.} \quad L_{13}(x)^2=1-c_{11}M_2(x)^2M_3(x)^2,$$

where c_{11} is a complex constant such that $c_{11}(\alpha_2+\alpha_3)=0$. Use $\sqrt{1-c_{11}M_2(x)^2M_3(x)^2}$ to denote the fixed branch of the square-root function of $1-c_{11}M_2(x)^2M_3(x)^2$ such that $\sqrt{1}=1$. The above gives

$$L_{13}(x)=\pm\sqrt{1-c_{11}M_2(x)^2M_3(x)^2}.$$

Also, in view of $a_{23}(u,x,y)\neq 1-a_{22}(u,x,y)a_{33}(u,x,y)$, we may take the branch with minus sign, so we get, by (7.11), when $c_{10}\neq 0$,

$$a_{23}(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y)\sqrt{1-c_{11}M_2(y)^2M_3(y)^2}-\sqrt{1-c_{11}M_2(x)^2M_3(x)^2}. \tag{7.12}$$

We note that (E20) and (E21) hold for this solution of a_{23}, and that for this solution of a_{23}, one can easily check the validity of (E4).

Note. The assumptions $a_{23}(u,x,y)\neq 0$, $a_{22}(u,x,y)a_{33}(u,x,y)\neq 1$ and $a_{23}(u,x,y)\neq 1-a_{22}(u,x,y)a_{33}(u,x,y)$ are equivalent to that $M_2(x)M_3(x)$ is not a constant when $\alpha_2+\alpha_3=0$. This can be seen as follows.

By Lemma 3.1 we see that $a_{22}(u,x,y)a_{33}(u,x,y)\neq 1$ if and only if either $\alpha_2+\alpha_3\neq 0$, or $M_2(x)M_3(x)$ is not a constant. For $\alpha_2+\alpha_3\neq 0$, we have $c_{11}=0$, so by (7.12) we see that $a_{23}(u,x,y)\neq 0$ and $a_{23}(u,x,y)\neq 1-a_{22}(u,x,y)a_{33}(u,x,y)$, which coincide with the assumptions on $a_{23}(u,x,y)$. If $\alpha_2+\alpha_3=0$, then by (7.12) we see that

$$a_{23}(u,x,y)=0$$

if and only if

$$\frac{\sqrt{1-c_{11}M_2(y)^2M_3(y)^2}}{M_2(y)M_3(y)}=\frac{\sqrt{1-c_{11}M_2(x)^2M_3(x)^2}}{M_2(x)M_3(x)},$$

or, for some constant c_{12},

$$\frac{\sqrt{1-c_{11}M_2(x)^2M_3(x)^2}}{M_2(x)M_3(x)}=c_{12},$$

i.e.

$$\bigl(c_{12}^2+c_{11}\bigr)M_2(x)^2M_3(x)^2=1,$$

which implies that $M_2(x)M_3(x)$ is a constant. Also, when $\alpha_2+\alpha_3=0$, by (7.12) we see that

$$a_{23}(u,x,y)=1-a_{22}(u,x,y)a_{33}(u,x,y)$$

if and only if

$$\frac{1+\sqrt{1-c_{11}M_2(y)^2M_3(y)^2}}{M_2(y)M_3(y)} = \frac{1+\sqrt{1-c_{11}M_2(x)^2M_3(x)^2}}{M_2(x)M_3(x)},$$

or, for some constant c_{13},

$$\frac{1+\sqrt{1-c_{11}M_2(x)^2M_3(x)^2}}{M_2(x)M_3(x)} = c_{13},$$

i.e.

$$\left(c_{13}^2+c_{11}\right)M_2(x)^2M_3(x)^2 = 2c_{13},$$

which also implies that $M_2(x)M_3(x)$ is a constant when $c_{11} \neq 0$.

Now to complete the investigation, we only need to give the general solution for a_{23} of Eq. (E4) when $c_{10}=0$. For this, we need to use the following Lemma 7.1. Let

$$g_1(u,x,y)=1, \quad g_2(u,x,y)=a_{22}(u,x,y)a_{33}(u,x,y),$$

in Lemma 7.1, then by Lemma 3.1 we have

$$g_2(u,b,b)=g_1(u,b,b)=1 \quad \text{if and only if} \quad \alpha_2+\alpha_3=0. \tag{7.13}$$

Also, when $\alpha_2+\alpha_3=0$, we have

$$g_2(u-2b,x,b)=g_2(u-2b,x,b)g_2(-u+3b,b,b)=g_2(b,x,b),$$

which together with Lemma 7.1 and Lemma 3.1 gives, for $\alpha_2+\alpha_3=0$,

$$a_{23}(u,x,y)=M(x)-a_{22}(u,x,y)a_{33}(u,x,y)M(y)+\lambda u M_2(x)M_3(x); \tag{7.14}$$

also, for $\alpha_2+\alpha_3\neq 0$, Lemma 7.1 together with Lemma 3.1 gives

$$a_{23}(u,x,y)=M(x)-a_{22}(u,x,y)a_{33}(u,x,y)M(y)+\lambda\left[1-\exp\left((\alpha_2+\alpha_3)u\right)\right]M_2(x)M_3(x), \tag{7.15}$$

where λ is a complex constant, $M(x)$ is a meromorphic function. Combining (7.14) and (7.15), we can conclude that

$$\begin{aligned} a_{23}(u,x,y) = {} & M(x)-a_{22}(u,x,y)a_{33}(u,x,y)M(y) \\ & +\lambda\left[1-\exp\left((\alpha_2+\alpha_3)u\right)\right]M_2(x)M_3(x)+\lambda\delta u M_2(x)M_3(x), \end{aligned} \tag{7.16}$$

where λ is a complex constant, and

$$\delta = \begin{cases} 1, & \alpha_2+\alpha_3=0, \\ 0, & \alpha_2+\alpha_3\neq 0. \end{cases} \tag{7.17}$$

Note. The assumptions $a_{23}(u,x,y)\neq 0$, $a_{22}(u,x,y)a_{33}(u,x,y)\neq 1$ and $a_{23}(u,x,y)\neq 1-a_{22}(u,x,y)a_{33}(u,x,y)$ are equivalent to one of the following two conditions: (1) if $\alpha_2+\alpha_3\neq 0$, then $M(x)\neq -\lambda M_2(x)M_3(x)$, $M(x)\neq 1-\lambda M_2(x)M_3(x)$, or (2) if $\alpha_2+\alpha_3=0$, then $M_2(x)M_3(x)$ is not a constant, and for $\lambda=0$, both $\frac{M(x)}{M_2(x)M_3(x)}$ and $\frac{M(x)-1}{M_2(x)M_3(x)}$ are not constants. This can be seen as follows.

Firstly, we note by Lemma 3.1 that $a_{22}(u,x,y)a_{33}(u,x,y)\neq 1$ is equivalent to $M_2(x)M_3(x)$ is not a constant when $\alpha_2+\alpha_3=0$. Secondly, when $\alpha_2+\alpha_3=0$, by (7.16) and (7.17), we see that $a_{23}(u,x,y)=0$ if and only if

$$\frac{M(x)}{M_2(x)M_3(x)}-\frac{M(y)}{M_2(y)M_3(y)}=\lambda u;$$

and $a_{23}(u,x,y)=1-a_{22}(u,x,y)a_{33}(u,x,y)$ if and only if

$$\frac{M(x)-1}{M_2(x)M_3(x)}-\frac{M(y)-1}{M_2(y)M_3(y)}=-\lambda u.$$

And thirdly, when $\alpha_2+\alpha_3\neq 0$, we note that by (7.16) and (7.17), $a_{23}(u,x,y)=0$ if and only if

$$\frac{M(x)}{M_2(x)M_3(x)}+\lambda = e^{(\alpha_2+\alpha_3)u}\left[\frac{M(y)}{M_2(y)M_3(y)}+\lambda\right],$$

i.e.

$$M(x) = -\lambda M_2(x)M_3(x);$$

and $a_{23}(u,x,y) = 1 - a_{22}(u,x,y)a_{33}(u,x,y)$ if and only if

$$\frac{M(x)-1}{M_2(x)M_3(x)} + \lambda = e^{(\alpha_2+\alpha_3)u}\left[\frac{M(y)-1}{M_2(y)M_3(y)} + \lambda\right],$$

i.e.

$$M(x) = 1 - \lambda M_2(x)M_3(x).$$

Gathering together the above, we get the above note.

Now, using (7.1) and (7.6) for the substitutions of a_{12}, a_{13}, a_{34} and a_{14}, then using (E4) and Lemma 3.1 for cancellation and simplification, we can obtain from (E16) that $c_{10}\alpha_2 = 0$. Therefore, the combination of (7.1), (7.3), (7.6), (7.12) and (7.16) proves Theorem 3.

Finally, we come to prove the following

Lemma 7.1. *Suppose that $g_i(u,x,y)$, $i = 1,2$, are given functions satisfying*

$$g_i(u+v,x,z) = g_i(u,x,y)g_i(v,y,z), \quad g_i(-u,y,x) = g_i(u,x,y)^{-1}. \tag{7.18}$$

Then the general solution for the function equation

$$f(u+v,x,z) = g_1(v,y,z)f(u,x,y) + g_2(u,x,y)f(v,y,z), \tag{7.19}$$

with unknown f can be expressed as

$$f(u,x,y) = \begin{cases} g_1(u-b,b,y)M(x) - g_1(-b,b,y)g_2(u,x,y)M(y) + \lambda_1 g_1(u+b,b,y)g_2(b,x,b) \\ \quad - \lambda_1 g_1(b,b,y)g_2(u+b,x,b), \quad \text{if } g_1(u,b,b) \neq g_2(u,b,b), \\ g_1(u-b,b,y)M(x) - g_1(-b,b,y)g_2(u,x,y)M(y) + \lambda_2 u g_1(u+b,b,y)g_2(b,x,b) \\ \quad - c(b)g_1(u-5b,b,y)g_2(b,x,b) + c(b)g_1(-2b,b,y)g_2(u-2b,x,b), \\ \quad\quad \text{if } g_1(u,b,b) = g_2(u,b,b), \end{cases}$$

where b is a fixed constant, λ_1, λ_2 are arbitrary constants, $c(b)$ is a constant depending at most on b, $M(x)$ is a meromorphic function.

Proof. Take a fixed point (b,b,b), which can ensure the definition of the involved functions below. Put

$$M(x) = f(b,x,b), \qquad N(x) = f(b,b,x), \qquad P(u) = f(u,b,b).$$

Then by (7.19) with $v = y = b$ we get

$$f(u+b,x,z) = g_1(b,b,z)f(u,x,b) + g_2(u,x,b)f(b,b,z).$$

This yields, with $u-b$ instead of u, and y instead of z,

$$\begin{aligned} f(u,x,y) &= g_1(b,b,y)f(u-b,x,b) + g_2(u-b,x,b)f(b,b,y) \\ &= g_1(b,b,y)f(u-b,x,b) + g_2(u-b,x,b)N(y). \end{aligned} \tag{7.20}$$

Again, (7.19) with $u = y = b$ gives

$$f(v+b,x,z) = g_1(v,b,z)f(b,x,b) + g_2(b,x,b)f(v,b,z),$$

and this yields, by replacing v and z by $u-2b$ and b respectively,

$$\begin{aligned} f(u-b,x,b) &= g_1(u-2b,b,b)f(b,x,b) + g_2(b,x,b)f(u-2b,b,b) \\ &= g_1(u-2b,b,b)M(x) + g_2(b,x,b)P(u-2b). \end{aligned}$$

Inserting this into (7.20), we get, by (7.18),

$$\begin{aligned} f(u,x,y) &= g_1(b,b,y)g_1(u-2b,b,b)M(x) + g_1(b,b,y)g_2(b,x,b)P(u-2b) + g_2(u-b,x,b)N(y) \\ &= g_1(u-b,b,y)M(x) + g_1(b,b,y)g_2(b,x,b)P(u-2b) + g_2(u-b,x,b)N(y). \end{aligned} \tag{7.21}$$

Using (7.21) to rewrite (7.19), we get

$$\begin{aligned} &g_1(u+v-b,b,z)M(x) + g_1(b,b,z)g_2(b,x,b)P(u+v-2b) + g_2(u+v-b,x,b)N(z) \\ &\quad = g_1(v,y,z)g_1(u-b,b,y)M(x) + g_1(v,y,z)g_1(b,b,y)g_2(b,x,b)P(u-2b) \\ &\qquad + g_1(v,y,z)g_2(u-b,x,b)N(y) + g_2(u,x,y)g_1(v-b,b,z)M(y) \\ &\qquad + g_2(u,x,y)g_1(b,b,z)g_2(b,y,b)P(v-2b) + g_2(u,x,y)g_2(v-b,y,b)N(z), \end{aligned}$$

i.e., by (7.18),

$$\begin{aligned} &g_1(b,b,z)g_2(b,x,b)P(u+v-2b) - g_1(v,y,z)g_1(b,b,y)g_2(b,x,b)P(u-2b) \\ &\qquad - g_2(u,x,y)g_1(b,b,z)g_2(b,y,b)P(v-2b) \\ &\quad = g_1(v,y,z)g_2(u-b,x,b)N(y) + g_2(u,x,y)g_1(v-b,b,z)M(y), \end{aligned}$$

or, also by (7.18),

$$\begin{aligned} &P(u+v-2b) - g_1(v,b,b)P(u-2b) - g_2(u,b,b)P(v-2b) \\ &\quad = g_1(v-b,y,b)g_2(u-2b,b,b)N(y) + g_2(u-b,b,y)g_1(v-2b,b,b)M(y). \end{aligned}$$

Multiplying both sides by $g_1(-v+3b,b,b)g_2(-u+3b,b,b)$, this can be written further as by (7.18),

$$\begin{aligned} &g_1(-v+3b,b,b)g_2(-u+3b,b,b)P(u+v-2b) \\ &\qquad - g_1(3b,b,b)g_2(-u+3b,b,b)P(u-2b) - g_1(-v+3b,b,b)g_2(3b,b,b)P(v-2b) \\ &\quad = g_1(2b,y,b)g_2(b,b,b)N(y) + g_1(b,b,b)g_2(2b,b,y)M(y). \end{aligned}$$

This implies that

$$g_1(2b,y,b)g_2(b,b,b)N(y) + g_1(b,b,b)g_2(2b,b,y)M(y) = c_{14}, \tag{7.22}$$

and

$$\begin{aligned} &g_1(-v+3b,b,b)g_2(-u+3b,b,b)P(u+v-2b) - g_1(3b,b,b)g_2(-u+3b,b,b)P(u-2b) \\ &\quad - g_1(-v+3b,b,b)g_2(3b,b,b)P(v-2b) = c_{14}, \end{aligned} \tag{7.23}$$

where c_{14} is a complex constant depending at most on b. The combination of (7.22) and (7.18) implies

$$\begin{aligned} N(y) &= c_{14}g_1(2b,y,b)^{-1}g_2(b,b,b)^{-1} \\ &\qquad - g_1(2b,y,b)^{-1}g_2(b,b,b)^{-1}g_1(b,b,b)g_2(2b,b,y)M(y) \\ &= c_{14}g_1(-2b,b,y)g_2(-b,b,b) - g_1(-2b,b,y)g_2(-b,b,b)g_1(b,b,b)g_2(2b,b,y)M(y) \\ &= c_{14}g_1(-2b,b,y)g_2(-b,b,b) - g_1(-b,b,y)g_2(b,b,y)M(y). \end{aligned} \tag{7.24}$$

Next, we separate two cases according to $g_1(u,b,b) = g_2(u,b,b)$ or not to consider (7.23). When $g_1(u,b,b) \neq g_2(u,b,b)$, using (7.18), we can write (7.23) as

$$P(u+v-2b) = g_1(v,b,b)P(u-2b) + g_2(u,b,b)P(v-2b) + c_{14}g_1(v-3b,b,b)g_2(u-3b,b,b). \tag{7.25}$$

For any solution P of (7.25), we also have

$$P(u+v-2b) = g_1(u,b,b)P(v-2b) + g_2(v,b,b)P(u-2b) + c_{14}g_1(u-3b,b,b)g_2(v-3b,b,b),$$

so there holds

$$\begin{aligned} &g_1(v,b,b)P(u-2b) + g_2(u,b,b)P(v-2b) + c_{14}g_1(v-3b,b,b)g_2(u-3b,b,b) \\ &\quad = g_1(u,b,b)P(v-2b) + g_2(v,b,b)P(u-2b) + c_{14}g_1(u-3b,b,b)g_2(v-3b,b,b), \end{aligned}$$

or

$$\begin{aligned}
&\big[g_1(v,b,b)-g_2(v,b,b)\big]P(u-2b)+c_{14}g_1(v-3b,b,b)g_2(u-3b,b,b)\\
&\qquad -c_{14}g_1(-3b,b,b)g_2(-3b,b,b)g_2(u,b,b)g_2(v,b,b)\\
&\quad =\big[g_1(u,b,b)-g_2(u,b,b)\big]P(v-2b)+c_{14}g_1(u-3b,b,b)g_2(v-3b,b,b)\\
&\qquad -c_{14}g_1(-3b,b,b)g_2(-3b,b,b)g_2(u,b,b)g_2(v,b,b),
\end{aligned}$$

i.e.

$$\begin{aligned}
&\big[g_1(v,b,b)-g_2(v,b,b)\big]\big[P(u-2b)+c_{14}g_1(-3b,b,b)g_2(-3b,b,b)g_2(u,b,b)\big]\\
&\quad =\big[g_1(u,b,b)-g_2(u,b,b)\big]\big[P(v-2b)+c_{14}g_1(-3b,b,b)g_2(-3b,b,b)g_2(v,b,b)\big].
\end{aligned}$$

This implies that

$$P(u-2b)=\lambda_1\big[g_1(u,b,b)-g_2(u,b,b)\big]-c_{14}g_1(-3b,b,b)g_2(-3b,b,b)g_2(u,b,b),$$

where λ_1 is a constant. Substituting this and (7.24) into (7.21), we get for $g_1(u,b,b)\neq g_2(u,b,b)$,

$$\begin{aligned}
f(u,x,y)&=g_1(u-b,b,y)M(x)+\lambda_1 g_1(b,b,y)g_2(b,x,b)\big[g_1(u,b,b)-g_2(u,b,b)\big]\\
&\quad -c_{14}g_1(b,b,y)g_2(b,x,b)g_1(-3b,b,b)g_2(-3b,b,b)g_2(u,b,b)\\
&\quad +c_{14}g_2(u-b,x,b)g_1(-2b,b,y)g_2(-b,b,b)\\
&\quad -g_2(u-b,x,b)g_1(-b,b,y)g_2(b,b,y)M(y)\\
&=g_1(u-b,b,y)M(x)+\lambda_1 g_1(u,b,b)g_1(b,b,y)g_2(b,x,b)\\
&\quad -\lambda_1 g_1(b,b,y)g_2(b,x,b)g_2(u,b,b)\\
&\quad -c_{14}g_1(b,b,y)g_1(-3b,b,b)g_2(b,x,b)g_2(u-3b,b,b)\\
&\quad +c_{14}g_1(-2b,b,y)g_2(u-b,x,b)g_2(-b,b,b)\\
&\quad -g_1(-b,b,y)g_2(u-b,x,b)g_2(b,b,y)M(y)\\
&=g_1(u-b,b,y)M(x)-g_1(-b,b,y)g_2(u,x,y)M(y)\\
&\quad +\lambda_1 g_1(u+b,b,y)g_2(b,x,b)-\lambda_1 g_1(b,b,y)g_2(u+b,x,b).
\end{aligned}\tag{7.26}$$

When $g_1(u,b,b)=g_2(u,b,b)$, (7.23) can be written as, by using (7.18),

$$g_1(-u-v,b,b)P(u+v-2b)-g_1(-u,b,b)P(u-2b)-g_1(-v,b,b)P(v-2b)=c_{14}g_1(-6b,b,b).$$

Putting

$$h(u)=g_1(-u,b,b)P(u-2b),$$

the above can be written further as

$$h(u+v)-h(u)-h(v)=c_{14}g_1(-6b,b,b).$$

This implies that

$$h(u)=\lambda_2 u-c_{14}g_1(-6b,b,b),$$

so, by (7.18),

$$P(u-2b)=\lambda_2 u g_1(u,b,b)-c_{14}g_1(u-6b,b,b),\tag{7.27}$$

where λ_2 is a complex constant. Substituting (7.24) and (7.27) into (7.21), we get

$$\begin{aligned}
f(u,x,y) &= g_1(u-b,b,y)M(x) + \lambda_2 u g_1(b,b,y)g_1(u,b,b)g_2(b,x,b) \\
&\quad - c_{14}g_1(b,b,y)g_1(u-6b,b,b)g_2(b,x,b) + c_{14}g_2(u-b,x,b)g_2(-b,b,b)g_1(-2b,b,y) \\
&\quad - g_2(u-b,x,b)g_1(-b,b,y)g_2(b,b,y)M(y) \\
&= g_1(u-b,b,y)M(x) - g_1(-b,b,y)g_2(u,x,y)M(y) \\
&\quad + \lambda_2 u g_1(u+b,b,y)g_2(b,x,b) - c_{14}g_1(u-5b,b,y)g_2(b,x,b) \\
&\quad + c_{14}g_1(-2b,b,y)g_2(u-2b,x,b). \qquad (7.28)
\end{aligned}$$

The combination of (7.26) and (7.28) completes the proof of Lemma 7.1. □

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